

Calculus, Bachelor in Global Governance. 2026/27 First semester (until October 7)

Yoh Tanimoto

Dipartimento di Matematica, Università di Roma Tor Vergata
Via della Ricerca Scientifica 1, I-00133 Roma, Italy
email: hoyt@mat.uniroma2.it

2026.09.14: Introduction of the course, Functions

Basic information

Lecture notes (or exercises): GG $\rightarrow$ [Calculus](#) $\rightarrow$ Teaching material,

or Yoh Tanimoto $\rightarrow$ Teaching $\rightarrow$ [2026 GG](#) (From October 8, Prof. Cipolloni will give lessons)

Textbook: Hoffmann, Bradley, Sobiecki, Price “Applied Calculus for Business, Economics, and the Social and Life Sciences”

Office hours: Wed. 14:00–15:00 (or by appointment)

Why study calculus?

You will study law, economics, history, English, logic, statistics, group interaction, etc. In some of these disciplines, numbers will play an important role.

Example: Tax rate and government revenue

Imagine that the government imposes a tax on business activities. If the tax rate is lower, there are more activities. If the tax rate is higher, there are fewer activities.

τ : tax rate.

$B(\tau)$: business activity at tax rate τ .

What happens to the tax revenue if τ rises? $\rightarrow$ Model.

τ	0	20	40	60	80	100
$B(\tau)$	100	80	60	40	20	0
tax revenue $\tau \cdot B(\tau)/100$	0	16	24	24	16	0

The tax revenue of the government is $B(\tau) \cdot \tau/100$.

A higher tax rate does not mean higher revenue.

Mathematical models help us understand what happens when we change something... if the models reflect the real world correctly.

Example: Minimum wage and employment rate

Let us consider what happens if the government imposes a minimum wage.

W : minimum wage.

$L(W)$: labour demand (employment).

What happens if W rises in one state, but not in another state?

Assume that the labour market is in “perfect competition.” If the minimum wage were raised, companies would have less profit from workers with lower wage. So we expect less employment.

Data: New Jersey, 1992. \$4.25/h $\rightarrow$ \$5.05/h.¹ Slight increase in employment at fast-food restaurants. (Later study: small negative effect on employment in general.²)

To understand how society works:

- need to make correct assumptions;
- need to understand how the model works;
- need to check whether the model matches reality.

In Calculus, we study tools to understand, use, and analyze models correctly: functions, derivatives, integrals, graphs, differential equations.

Functions (Textbook section 1.1)

A function is a rule that assigns to each number x in A (“domain”) one (and only one) number y in B (“codomain”, sometimes “range”).

$$y = f(x).$$

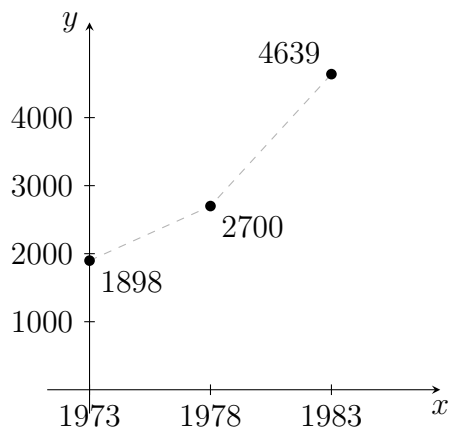
¹D. Card and A. B. Krueger, “Minimum Wages and Employment: A Case Study of the Fast-Food Industry in New Jersey and Pennsylvania,” *American Economic Review* **84** (1994), 772–793. [nber.org/papers/w4509](https://www.nber.org/papers/w4509).

²D. Neumark and P. Shirley, “Myth or Measurement: What Does the New Minimum Wage Research Say about Minimum Wages and Job Loss in the United States?,” *Industrial Relations* **61** (2022), 384–417. [nber.org/papers/w28388](https://www.nber.org/papers/w28388).

An example from data: average tuition fees, 4-year college.

x (year)	1973	1978	1983
$f(x)$ (US dollar)	1898	2700	4639

The domain here is a finite set of years, so the graph is a set of separate points, not a curve.



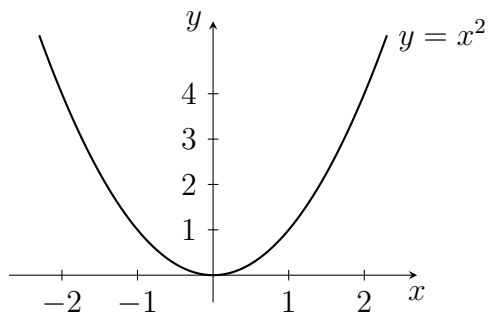
An example from a model:

$$y = f(x) = x^2.$$

If we take concrete numbers,

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2
$f(x)$	4	1	$\frac{1}{4}$	0	$\frac{1}{9}$	$\frac{1}{4}$	1	4

Its graph can be drawn as follows.

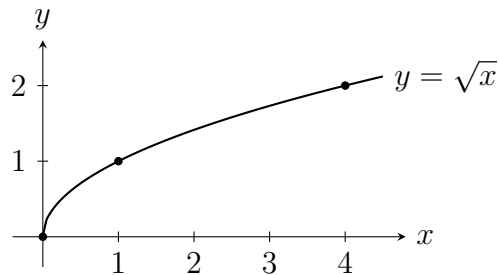


Examples

$$f(x) = \sqrt{x}$$

is defined for $x \geq 0$.

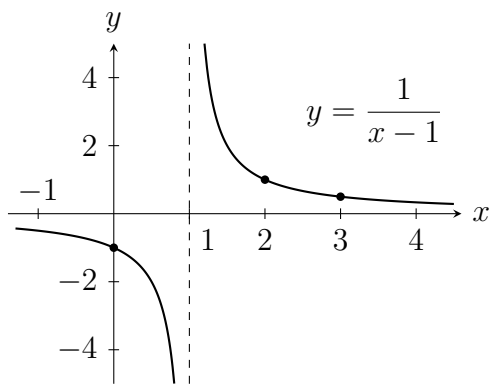
$$f(0) = 0, \quad f(1) = 1, \quad f(4) = 2.$$



$$f(x) = \frac{1}{x-1}$$

is defined for $x \neq 1$.

$$f(0) = -1, \quad f(2) = 1, \quad f(3) = \frac{1}{2}.$$

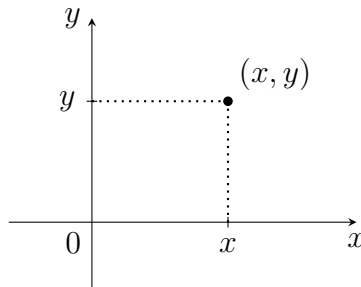


Graph of a function (section 1.2)

If we have a function f , we can plot its graph.

Take the example above.

In general, consider a (data) point (x, y) , where x, y are numbers. We can plot that point on a plane. x, y are called the coordinates of the point (x, y) .



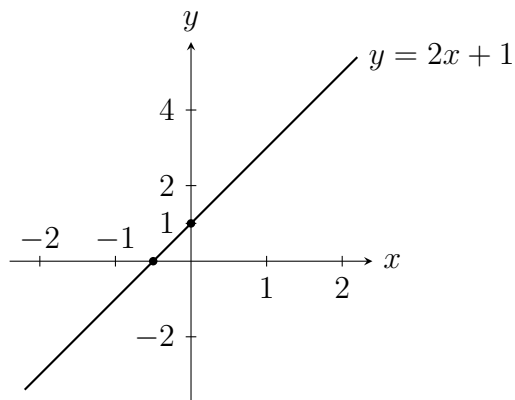
The graph of a function f is the collection of the points

$$(x, f(x)),$$

where x belongs to the domain of f .

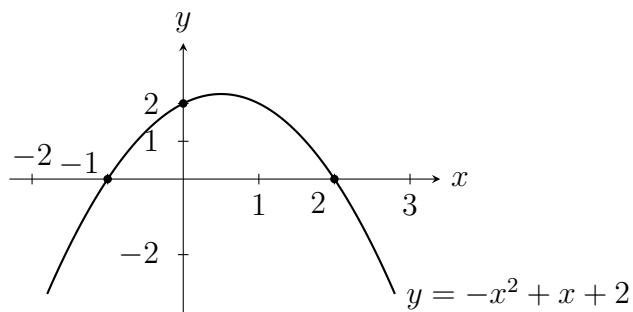
Example.

$$f(x) = 2x + 1.$$



Another sketch in the source:

$$f(x) = -x^2 + x + 2.$$



When does the graph intersect the x -axis and y -axis?

The **y -intercept** is $(0, f(0))$. The **x -intercept** is some $(x, f(x))$ such that $f(x) = 0$.

Example.

$$f(x) = 2x + 1.$$

y -intercept: $(0, 1)$.

x -intercept:

$$2x + 1 = 0 \implies x = -\frac{1}{2},$$

so $(-\frac{1}{2}, 0)$.

Example.

$$f(x) = -x^2 + x + 2.$$

y -intercept: $(0, 2)$.

x -intercepts:

$$-x^2 + x + 2 = 0 \iff x^2 - x - 2 = 0 \iff (x + 1)(x - 2) = 0,$$

so $(-1, 0)$ and $(2, 0)$.

Example.

$$f(x) = \frac{x + 1}{x - 2}.$$

This is defined for $x \neq 2$.

y -intercept:

$$f(0) = \frac{0 + 1}{0 - 2} = -\frac{1}{2},$$

so $(0, -\frac{1}{2})$.

x -intercept: a fraction is 0 exactly when its numerator is 0 (and its denominator is not), so

$$\frac{x + 1}{x - 2} = 0 \iff x + 1 = 0 \iff x = -1,$$

so $(-1, 0)$.

2026.09.15: Examples of functions

Linear function

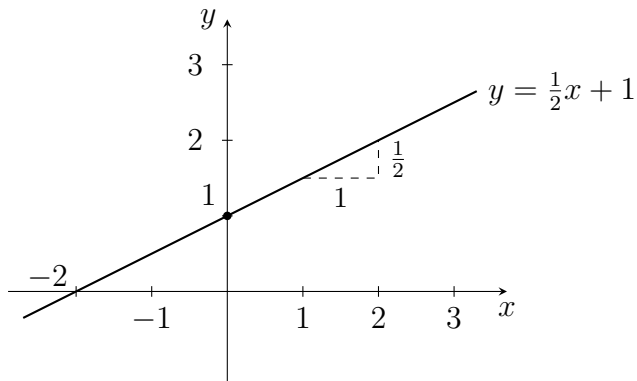
$$f(x) = ax + b,$$

here a, b are constants, which do not change when x changes.

Domain: all real numbers, $(-\infty, \infty)$.

The y -intercept is b ; the slope is a .

The following is the graph of $f(x) = \frac{1}{2}x + 1$.



Quadratic function

Special forms:

$$f(x) = ax^2,$$

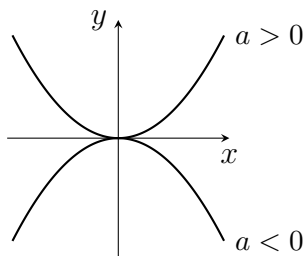
with sketches for $a > 0$ and $a < 0$. Domain: all real numbers, $(-\infty, \infty)$.

$$f(x) = a(x - b)^2,$$

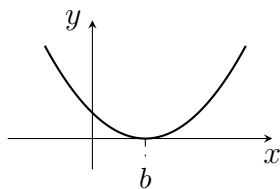
with vertex at $x = b$;

$$f(x) = x^2 + c,$$

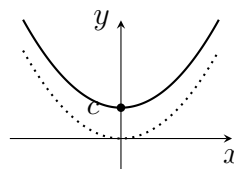
with vertical shift c .



$$f(x) = ax^2$$



$$f(x) = a(x - b)^2$$



$$f(x) = x^2 + c$$

In general,

$$f(x) = ax^2 + bx + c, \quad a \neq 0.$$

If $a > 0$ or $a < 0$, it is a parabola.

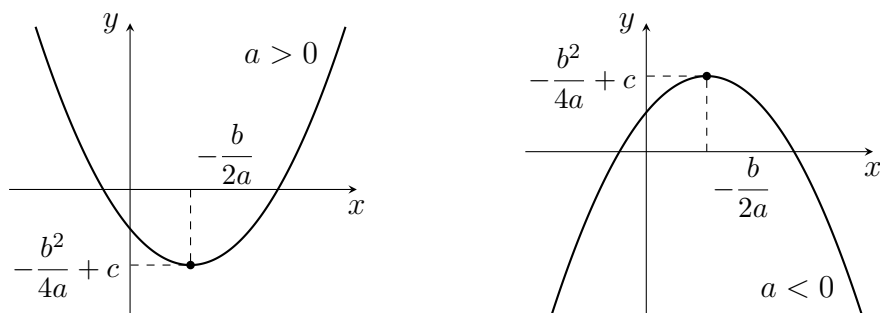
Completing the square:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a \left(x^2 + \frac{b}{a}x \right) + c \end{aligned}$$

$$\begin{aligned}
&= a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} \right) + c \\
&= a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c.
\end{aligned}$$

Recall:

$$(x + A)^2 = x^2 + 2Ax + A^2.$$



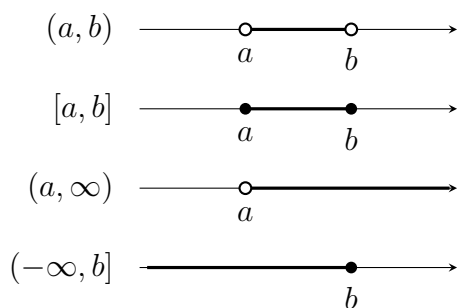
Intervals

(a, b) : numbers larger than a and smaller than b ,

$[a, b]$: numbers not smaller than a and not larger than b ,

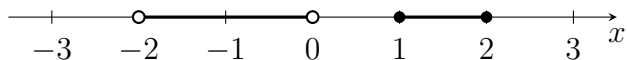
(a, ∞) : numbers larger than a ,

$(-\infty, b]$: numbers not larger than b .



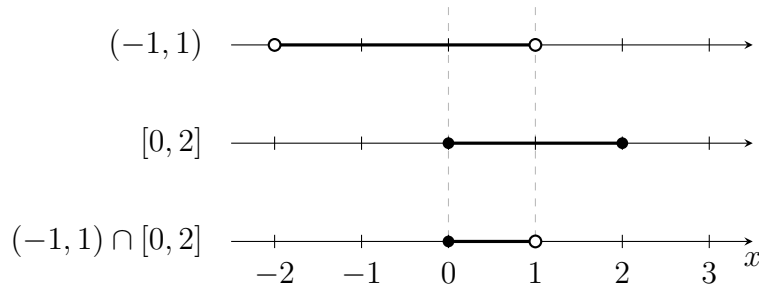
Union of intervals is the “set” of points contained either in one or the other:

$$(-2, 0) \cup [1, 2].$$



Intersection of intervals is the “set” of points contained in both:

$$(-1, 1) \cap [0, 2] = [0, 1).$$

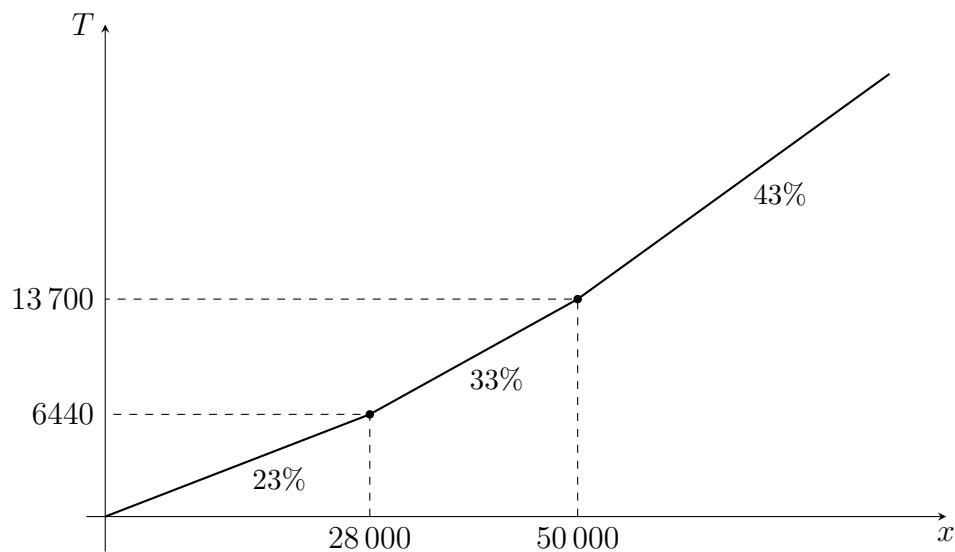


The left endpoint 0 is *included*, because 0 belongs to $[0, 2]$ and lies inside $(-2, 1)$. The right endpoint 1 is *excluded*, because 1 does not belong to $(-1, 1)$.

A real-world example: Italian income tax

Let x be the income (in euro). The income tax $T(x)$ is determined as follows: the rate is 23% on the part of the income up to 28 000, 33% on the part between 28 000 and 50 000, and 43% on the part above 50 000.³

$$T(x) = \begin{cases} 0.23x, & 0 \leq x \leq 28\,000, \\ 6440 + 0.33(x - 28\,000), & 28\,000 < x \leq 50\,000, \\ 13\,700 + 0.43(x - 50\,000), & x > 50\,000. \end{cases}$$



³Agenzia delle Entrate, agenziaentrate.gov.it.

Operations on functions

If $f(x)$ and $g(x)$ are two functions, then:

- $f(x) + g(x)$ is another function;
- $f(x) - g(x)$ is another function;
- $f(x)g(x)$ is another function;
- $\frac{f(x)}{g(x)}$ is another function, defined for x such that $g(x) \neq 0$;
- $f(g(x))$ is another function, if $g(x)$ is in the domain of f (composition).

Examples.

- $f(x) = x, g(x) = a$:

$$f(x) + g(x) = x + a, \quad f(x)g(x) = ax.$$

- $f(x) = x, g(x) = x^2$:

$$f(x) + g(x) = x + x^2, \quad f(x)g(x) = x^3.$$

- $f(x) = x + 2, g(x) = x^2$:

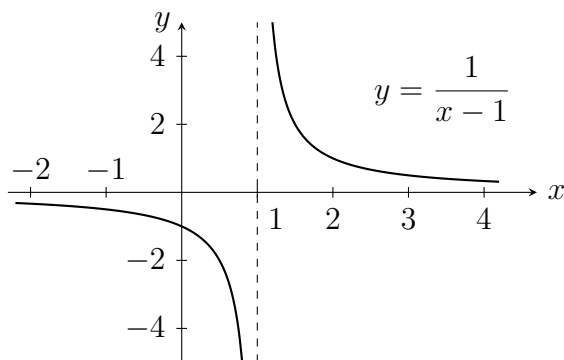
$$f(g(x)) = x^2 + 2, \quad g(f(x)) = (x + 2)^2.$$

Rational function

$$f(x) = \frac{1}{x - 1}.$$

Domain:

$$(-\infty, 1) \cup (1, \infty).$$



Another rational function:

$$f(x) = \frac{x^3 + x^2 + 4}{x^2 - 3x}.$$

Root

For $x \geq 0$, there is y such that

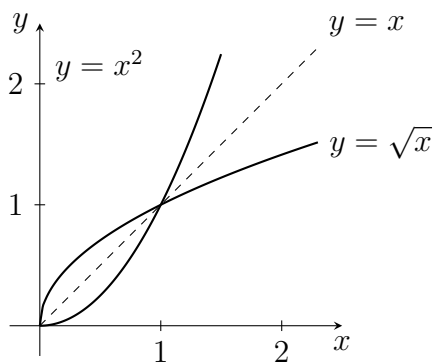
$$y^2 = x.$$

y is called the square root; we write $\sqrt{x}$.

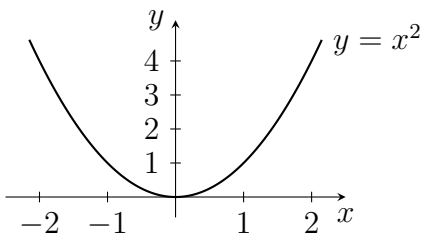
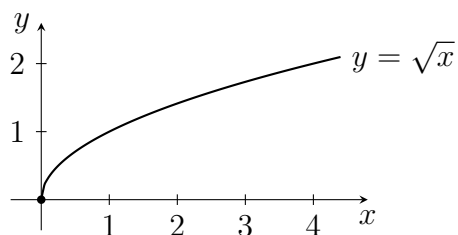
$$f(x) = \sqrt{x}$$

is a function defined on $[0, \infty)$.

$$\begin{aligned}\sqrt{1} &= 1, & \sqrt{0} &= 0, & \sqrt{4} &= 2, & \sqrt{9} &= 3, \\ \sqrt{2} &\approx 1.4142, & \sqrt{3} &\approx 1.73205 \dots\end{aligned}$$

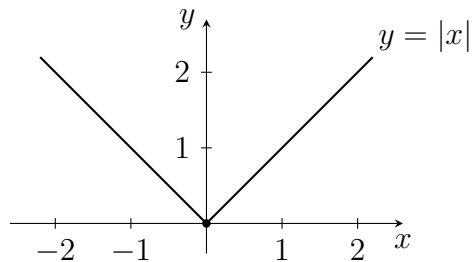


The graph of a function f is the collection of the points $(x, f(x))$, where x is in the domain of f .



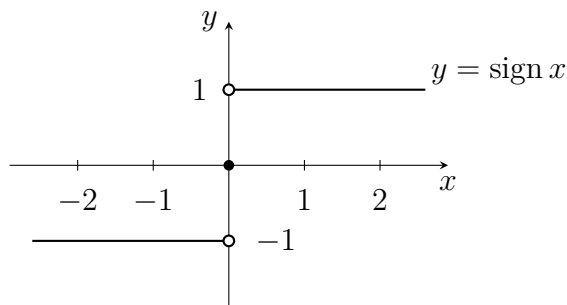
Absolute value

$$|x| = \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0. \end{cases}$$



Sign function

$$\operatorname{sign} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$



Exercises: domain of the following functions

1. $f(x) = 3x^4 + 4x$. Domain: $\mathbb{R}$.

2.

$$f(x) = \frac{x^2}{x-1}.$$

Need $x \neq 1$, so the domain is

$$(-\infty, 1) \cup (1, \infty).$$

3.

$$f(x) = \frac{x}{x^2 - 1}.$$

Need

$$x^2 - 1 \neq 0 \iff (x-1)(x+1) \neq 0 \iff x \neq -1, 1.$$

Domain:

$$(-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

4.

$$f(x) = \sqrt{x-1}.$$

Need

$$x-1 \geq 0 \iff x \geq 1,$$

hence the domain is $[1, \infty)$.

2026.09.17: Limits

Limits of functions (Textbook section 1.5)

We want to study the behaviour of f , what happens when x approaches certain values.

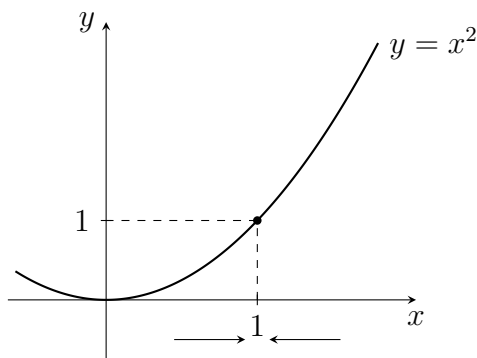
Examples. Consider

$$f(x) = x^2.$$

When x approaches 1, for example 1.1, 1.01, 1.001, $\dots$,

$$f(1.1) = 1.21, \quad f(1.01) = 1.0201, \quad f(1.001) = 1.002001,$$

while $f(1) = 1$.



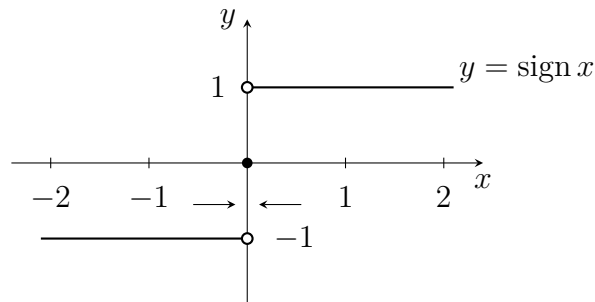
Consider

$$f(x) = \text{sign } x.$$

When x approaches 0, for example 0.1, 0.01, 0.001, $\dots$,

$$\text{sign}(0.1) = 1, \quad \text{sign}(0.01) = 1, \quad \text{sign}(0.001) = 1,$$

but $\text{sign}(0) = 0$. For negative values approaching 0, the sign is -1 .



Definition

If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c , from both sides, then we say that L is the limit of f as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = L.$$

Example.

$$f(x) = x^2, \quad c = 1,$$

so

$$\lim_{x \rightarrow 1} x^2 = 1.$$

For $f(x) = \text{sign } x$, when x approaches 0 from the right, $\text{sign } x$ remains 1; from the left, it remains -1 . Therefore

$$\lim_{x \rightarrow 0} \text{sign } x$$

does not exist.

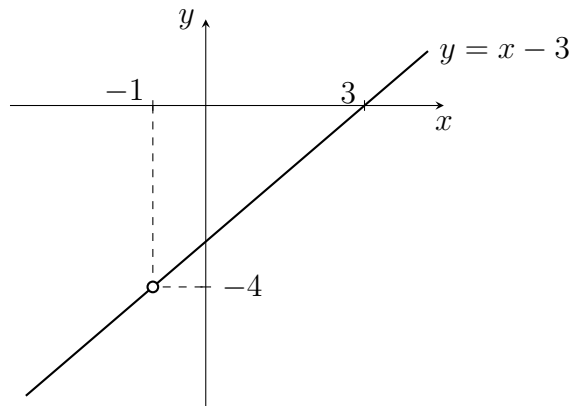
Example. Let

$$f(x) = \frac{x^2 - 2x - 3}{x + 1}, \quad x \neq -1.$$

When x approaches -1 ,

$$f(x) = \frac{x^2 - 2x - 3}{x + 1} = \frac{(x + 1)(x - 3)}{x + 1} = x - 3,$$

so $f(x)$ approaches -4 .



The graph is the line $y = x - 3$ with the point $(-1, -4)$ removed: $f(-1)$ is not defined, but the height of the graph still approaches -4 as x approaches -1 .

Example.

$$f(x) = \frac{1}{x}.$$

When x approaches 0 from the right, e.g. $0.1, 0.01, 0.001$,

$$f(0.1) = 10, \quad f(0.01) = 100, \quad f(0.001) = 1000.$$

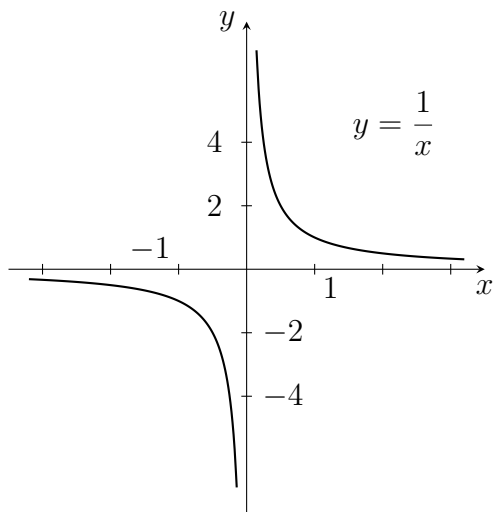
When x approaches 0 from the left, e.g. $-0.1, -0.01, -0.001$,

$$f(-0.1) = -10, \quad f(-0.01) = -100, \quad f(-0.001) = -1000.$$

So

$$\lim_{x \rightarrow 0} \frac{1}{x}$$

does not exist.



Geometrically, the limit exists when the height of the graph approaches L as x approaches c . This does not say much about $f(c)$ itself. Indeed, it can happen that

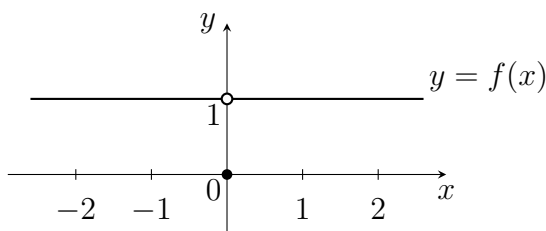
$$\lim_{x \rightarrow c} f(x) \neq f(c).$$

Example.

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \rightarrow 0} f(x) = 1, \quad \text{but} \quad f(0) = 0.$$



Rules of limits

Quotient form

We can define

$$\frac{f(x)}{g(x)}$$

if $g(x) \neq 0$. What happens to the limit of the quotient?

If

$$\lim_{x \rightarrow c} g(x) \neq 0,$$

then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}.$$

If $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} g(x) = 0$, the value $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ can vary. This case is called **indeterminate**.

For example, $\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0$ but $\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

Rules of limits

Assume that $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist. Then

$$\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x),$$

$$\lim_{x \rightarrow c} (f(x) - g(x)) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x),$$

$$\lim_{x \rightarrow c} (f(x)g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right),$$

and

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}, \quad \text{if } \lim_{x \rightarrow c} g(x) \neq 0.$$

Examples.

$$f(x) = x.$$

When x approaches c , $f(x) = x$ approaches c .

$$g(x) = k \quad (\text{constant}).$$

When x approaches c , $g(x) = k$ approaches k .

Example.

$$\lim_{x \rightarrow c} x^2 = \lim_{x \rightarrow c} (x \cdot x) = \left(\lim_{x \rightarrow c} x \right) \left(\lim_{x \rightarrow c} x \right) = c^2.$$

Also,

$$\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x),$$

and

$$\lim_{x \rightarrow c} (x^2 + 2x) = c^2 + 2c.$$

Example: rationalization

$$f(x) = \frac{\sqrt{x} - 1}{x - 1}.$$

When $x \rightarrow 1$, $x - 1 \rightarrow 0$. Numerically, values are close to 0.5. Indeed,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{(\sqrt{x})^2 - 1^2} \\ &= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{(\sqrt{x} + 1)(\sqrt{x} - 1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{1 + 1} = \frac{1}{2}. \end{aligned}$$

For an indeterminate limit, we cannot simply substitute x by c .

Example.

$$\lim_{x \rightarrow 1} \frac{3x^3 - 8}{x - 2}.$$

Since

$$\lim_{x \rightarrow 1} (x - 2) = -1 \neq 0,$$

we can substitute:

$$\lim_{x \rightarrow 1} \frac{3x^3 - 8}{x - 2} = \frac{-5}{-1} = 5.$$

Example.

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2}.$$

Since the denominator tends to 0, this is indeterminate. Factor:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)(x - 2)} \\ &= \lim_{x \rightarrow 1} \frac{x + 1}{x - 2} = \frac{2}{-1} = -2. \end{aligned}$$

If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

then f is called a polynomial, and

$$\lim_{x \rightarrow c} f(x) = f(c).$$

If

$$f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0},$$

then f is called a rational function, and

$$\lim_{x \rightarrow c} f(x) = f(c)$$

provided the denominator at c is nonzero.

2026.09.21: Infinite limits, one-sided limit

We studied the limit

$$\lim_{x \rightarrow c} f(x) = L,$$

meaning that $f(x)$ approaches L when x approaches c .

There are other cases, where f increases or decreases without bound. We say

$$\lim_{x \rightarrow c} f(x) = +\infty$$

if $f(x)$ increases without bound, and

$$\lim_{x \rightarrow c} f(x) = -\infty$$

if $f(x)$ decreases without bound.

Examples.

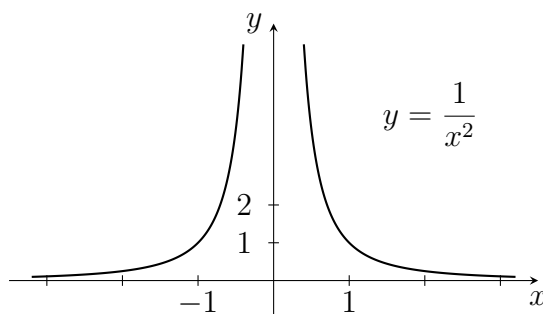
$$f(x) = \frac{1}{x^2}.$$

For instance,

$$f(0.1) = 100, \quad f(0.01) = 10000, \quad f(-0.001) = 1000000,$$

so

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$



Here both sides behave in the same way, so the limit is $+\infty$.

For

$$f(x) = \frac{1}{x},$$

positive values near 0 become large and positive, while negative values near 0 become large and negative, so

$$\lim_{x \rightarrow 0} \frac{1}{x}$$

does not exist.

If the values of f approach L as x increases without bound, we write

$$\lim_{x \rightarrow \infty} f(x) = L.$$

Similarly one may consider $x \rightarrow -\infty$.

Example.

$$f(x) = \frac{1}{x}.$$
$$f(10) = 0.1, \quad f(100) = 0.01,$$

and

$$f(-10) = -0.1, \quad f(-100) = -0.01.$$

Thus

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0, \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0.$$

Geometrically,

$$\lim_{x \rightarrow \infty} f(x) = L$$

exists if the graph of $f(x)$ approaches the line $y = L$, called a horizontal asymptote.

Similarly, if $k > 0$,

$$\lim_{x \rightarrow \infty} \frac{A}{x^k} = 0, \quad \lim_{x \rightarrow -\infty} \frac{A}{x^k} = 0,$$

where A is a constant.

Example. Find

$$\lim_{x \rightarrow \infty} \frac{x^2}{1 + x + 2x^2}.$$

Directly this looks like the indeterminate form ∞/∞ . Divide numerator and denominator by x^2 :

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2}{1 + x + 2x^2} &= \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x^2} + \frac{1}{x} + 2} \\ &= \frac{1}{2}. \end{aligned}$$

Algebraic limit rules also hold for $x \rightarrow \infty$ (when the relevant limits exist): sums, products, and quotients.

- $\lim_{x \rightarrow \infty} (f(x) + g(x)) = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$
- $\lim_{x \rightarrow \infty} (f(x) - g(x)) = \lim_{x \rightarrow \infty} f(x) - \lim_{x \rightarrow \infty} g(x)$
- $\lim_{x \rightarrow \infty} (f(x) \cdot g(x)) = (\lim_{x \rightarrow \infty} f(x)) \cdot (\lim_{x \rightarrow \infty} g(x))$
- $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow \infty} f(x)}{\lim_{x \rightarrow \infty} g(x)}$ (if $\lim_{x \rightarrow \infty} g(x) \neq 0$).

Exercise:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 1}{3x^2 - 5x + 2} &= \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x} + \frac{1}{x^2}}{3 - \frac{5}{x} + \frac{2}{x^2}} \\ &= \frac{2}{3}. \end{aligned}$$

Example: crop yield model

If a crop is planted in soil where the nitrogen level is N , the crop yield can be “modelled” (known by experience) by

$$Y(N) = \frac{AN}{B + N},$$

where A and B are positive constants (Michaelis–Menten).

$$\lim_{N \rightarrow \infty} Y(N) = \lim_{N \rightarrow \infty} \frac{A}{\frac{B}{N} + 1} = A.$$

Even if the nitrogen level increases, the crop yield does not increase unlimitedly.

Infinite-limit examples

$$f(x) = x, \quad \lim_{x \rightarrow \infty} x = \infty, \quad \lim_{x \rightarrow -\infty} x = -\infty.$$

$$f(x) = x^2, \quad \lim_{x \rightarrow \infty} x^2 = \infty, \quad \lim_{x \rightarrow -\infty} x^2 = \infty.$$

Exercise:

$$\lim_{x \rightarrow \infty} \frac{-x^3 + 2x + 1}{x - 3}.$$

This is an “indeterminate” form $-\infty/\infty$. We calculate

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{-x^3 + 2x + 1}{x - 3} &= \lim_{x \rightarrow \infty} -\frac{x^2 - 2 - \frac{1}{x}}{1 - \frac{3}{x}} \\ &= -\frac{\infty}{1} = -\infty. \end{aligned}$$

One-sided limits

Consider

$$\operatorname{sign} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

$$\lim_{x \rightarrow 0} \operatorname{sign} x$$

does not exist, but when x approaches 0 from the right, $\operatorname{sign} x$ remains 1. We write

$$\lim_{x \rightarrow 0^+} \operatorname{sign} x = 1.$$

More generally, if $f(x)$ approaches L when x approaches c from the right ($x > c$), we write

$$\lim_{x \rightarrow c^+} f(x) = L.$$

For approach from the left ($x < c$), we write

$$\lim_{x \rightarrow c^-} f(x) = L.$$

The two-sided limit exists if and only if both the one-sided limits exist.

Example. Tuition discount on the tuition fee at the University of Roma Tor Vergata depends on the ISEE x of their household (an indicator of income and assets).⁴

ISEE x	discount $f(x)$
$x < 26\,000$	100%
$26\,000 \leq x \leq 28\,000$	60%
$28\,000 < x \leq 30\,000$	55%
$x > 30\,000$	0%

$$\lim_{x \rightarrow 26000^+} f(x) = 60, \lim_{x \rightarrow 26000^-} f(x) = 100.$$

Example.

$$f(x) = \frac{1}{x}.$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

Example.

$$\lim_{x \rightarrow 4^+} \frac{x-2}{x-4}.$$

Here

$$\lim_{x \rightarrow 4^+} (x-2) = 2, \quad \lim_{x \rightarrow 4^+} \frac{1}{x-4} = \infty.$$

Thus

$$\lim_{x \rightarrow 4^+} \frac{x-2}{x-4} = \infty.$$

Exercises: Calculate

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2x - 3}{x^4 + 5x},$$

$$\lim_{x \rightarrow 0^-} \frac{x^3 + 2}{x}.$$

⁴Exemptions for a.a. 2024/25: studenti.uniroma2.it. Ordinary taxation

Integer part

The integer part is written $\lfloor x \rfloor$ in the notes (i.e. the floor function).

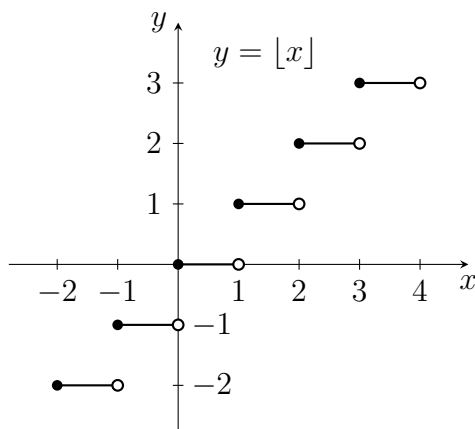
If $x = 1.4$, then

$$\lfloor x \rfloor = 1.$$

If $x = 3.14$, then

$$\lfloor 3.14 \rfloor = 3.$$

On each interval $[n, n + 1)$ the value is constantly n , so the graph is a staircase: the left endpoint of each step belongs to the graph, the right one does not.



At $x = 1$,

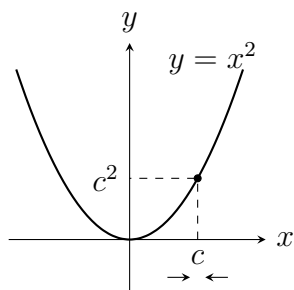
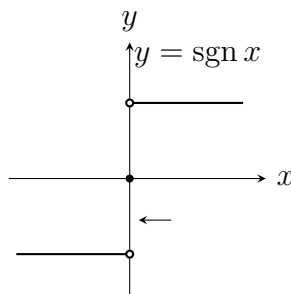
$$\lim_{x \rightarrow 1^+} \lfloor x \rfloor = 1, \quad \lim_{x \rightarrow 1^-} \lfloor x \rfloor = 0.$$

2026.09.22: Continuity

Continuity (Textbook section 1.6)

We have seen some functions and limits, for instance

$$\lim_{x \rightarrow c} x^2 = c^2, \quad \lim_{x \rightarrow 0^+} \operatorname{sgn} x = 1 \neq 0.$$



Definition. A function f is said to be *continuous* at c if

$$\lim_{x \rightarrow c} f(x) = f(c),$$

that is, c is in the domain of f , $\lim_{x \rightarrow c} f(x)$ exists, and it equals $f(c)$. Otherwise f is said to be *discontinuous* at c . The idea is that the graph of f is connected (unbroken) at c .

Examples.

- $f(x) = x$ is continuous: $\lim_{x \rightarrow c} x = c = f(c)$.

- $f(x) = x^2$ is continuous:

$$\lim_{x \rightarrow c} x^2 = \left(\lim_{x \rightarrow c} x \right) \left(\lim_{x \rightarrow c} x \right) = c \cdot c = c^2 = f(c).$$

- $f(x) = x^2 + x$ is continuous, using the rules of limits:

$$\lim_{x \rightarrow c} (x^2 + x) = \lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} x = c^2 + c = f(c).$$

- $f(x) = \frac{1}{x}$. If $c \neq 0$, $\frac{1}{x}$ is continuous at c :

$$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{\lim_{x \rightarrow c} x} = \frac{1}{c} = f(c).$$

If $c = 0$, $\frac{1}{x}$ is discontinuous at 0, because $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist.

These examples are all instances of the same principle: each rule of limits gives a rule of continuity.

Rules of continuity. Let f and g be continuous at c . Then

- $f(x) + g(x)$ and $f(x) - g(x)$ are continuous at c ;
- $f(x)g(x)$ is continuous at c ;
- $\frac{f(x)}{g(x)}$ is continuous at c , provided that $g(c) \neq 0$;
- if h is continuous at $f(c)$, then $h(f(x))$ is continuous at c (composition).

Each of these follows from the corresponding rule of limits. For the product, for instance,

$$\lim_{x \rightarrow c} f(x)g(x) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right) = f(c)g(c),$$

which is exactly the statement that fg is continuous at c .

In general, a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

(a polynomial) is continuous at every c . A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials (a rational function), is continuous at every c such that $q(c) \neq 0$.

Example.

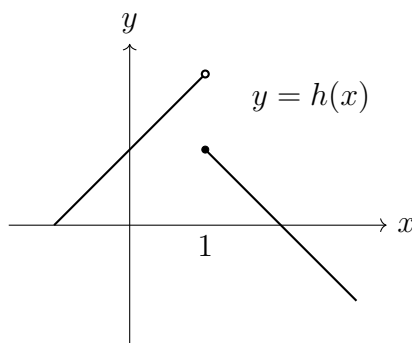
$$f(x) = \frac{x-1}{x-2}.$$

This f is continuous at $x = 3$. Indeed, $\lim_{x \rightarrow 3} (x-2) = 3-2 = 1 \neq 0$. Thus

$$\lim_{x \rightarrow 3} \frac{x-1}{x-2} = \frac{3-1}{3-2} = \frac{2}{1} = 2 = f(3).$$

Example.

$$h(x) = \begin{cases} x+1, & x < 1, \\ 2-x, & x \geq 1. \end{cases}$$



For $c \neq 1$, either $c < 1$ or $c \geq 1$. If $c < 1$, then $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} (x+1) = c+1 = h(c)$. If $c > 1$, then $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} (2-x) = 2-c = h(c)$.

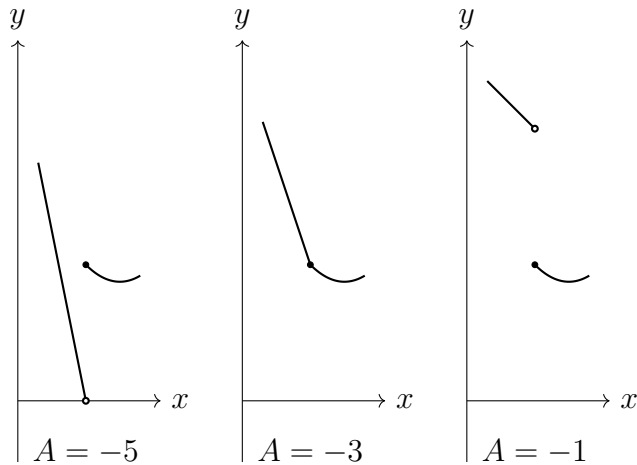
At $c = 1$,

$$\lim_{x \rightarrow 1^-} h(x) = \lim_{x \rightarrow 1^-} (x+1) = 1+1 = 2, \quad \lim_{x \rightarrow 1^+} h(x) = \lim_{x \rightarrow 1^+} (2-x) = 2-1 = 1.$$

Since these one-sided limits disagree, $\lim_{x \rightarrow 1} h(x)$ does not exist, so h is discontinuous at 1.

Exercise. For what value of A is the following f continuous for all x ?

$$f(x) = \begin{cases} Ax+5, & x < 1, \\ x^2-3x+4, & x \geq 1. \end{cases}$$



The picture above shows three different values of A : for $A = -5$ or $A = -1$ the graph jumps at $x = 1$, while for $A = -3$ the two pieces meet. f is already continuous for $x < 1$ and for $x > 1$. As for $x = 1$,

$$\lim_{x \rightarrow 1^-} (Ax + 5) = A + 5, \quad \lim_{x \rightarrow 1^+} (x^2 - 3x + 4) = 1^2 - 3 \cdot 1 + 4 = 2.$$

For f to be continuous at 1, these must agree: $A + 5 = 2$, so $A = -3$.

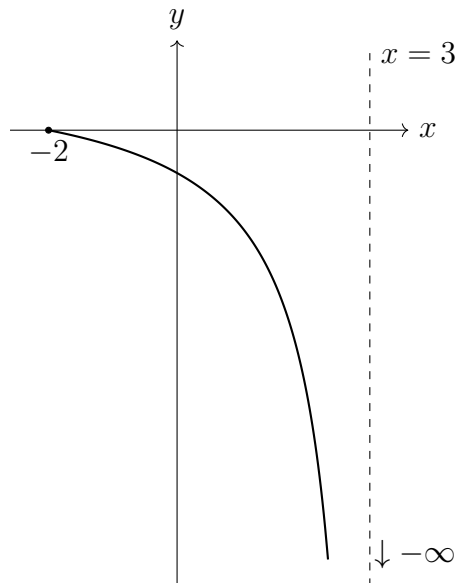
Continuity on a closed interval

For a function f defined on $[a, b]$, f is continuous on $[a, b]$ if it is continuous at each point of (a, b) and

$$f(a) = \lim_{x \rightarrow a^+} f(x), \quad f(b) = \lim_{x \rightarrow b^-} f(x).$$

Example.

$$f(x) = \frac{x+2}{x-3} \quad \text{on } (-2, 3).$$



f is a rational function, so it is continuous on $(-2, 3)$, but f is not continuous on $[-2, 3]$: $f(3)$ is not defined, and in fact

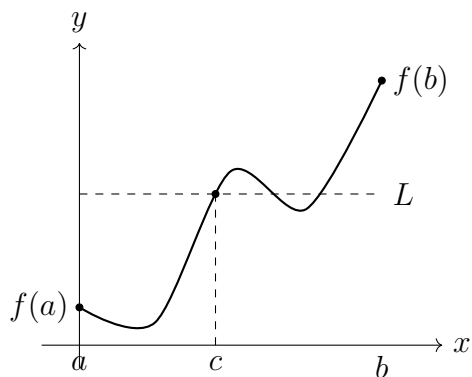
$$\lim_{x \rightarrow 3^-} \frac{x+2}{x-3} = -\infty.$$

Intermediate value theorem

Let f be a continuous function on $[a, b]$, and let L be a number between $f(a)$ and $f(b)$, that is,

$$f(a) < L < f(b) \quad \text{if } f(a) < f(b), \quad f(b) < L < f(a) \quad \text{if } f(b) < f(a).$$

Then there is $c \in (a, b)$ such that $f(c) = L$.



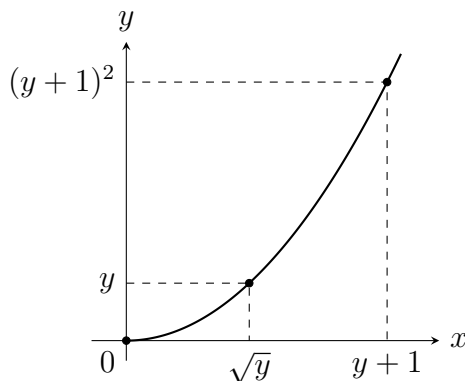
This is because the graph of f is “connected”: it rises from $f(a)$ to $f(b)$, so at some point c it must cross the height L . If you pour water into a bottle, at some point the level of the water reaches every height between the bottom and the top of the bottle.

Example. For $y > 0$, there is $x \geq 0$ such that $x^2 = y$.

Indeed, let $b = y + 1$. The function $f(x) = x^2$ is continuous on $[0, b]$, and

$$f(0) = 0 < y < (y + 1)^2 = f(b).$$

By the intermediate value theorem, there is $x \in (0, b)$ such that $x^2 = y$, that is, $x = \sqrt{y}$.



Exercise. Show that the equation

$$x^3 - x - 1 = 0$$

has a solution c with $1 < c < 2$.

Solution. Let $f(x) = x^3 - x - 1$. Since f is a polynomial, it is continuous on $[1, 2]$.

$$f(1) = 1 - 1 - 1 = -1 < 0, \quad f(2) = 8 - 2 - 1 = 5 > 0.$$

So $f(1) < 0 < f(2)$. By the intermediate value theorem (with $L = 0$), there is $c \in (1, 2)$ such that $f(c) = 0$, that is,

$$c^3 - c - 1 = 0.$$

2026.09.23: Exercises on limits and functions

Exercises: determine the domain of the function

1. $f(x) = \sqrt{1-x}$
2. $f(x) = \sqrt{x^2 - 5x + 4}$ (Solve the inequality $x^2 - 5x + 4 \geq 0$. This amounts to first solve $x^2 - 5x + 4 = 0$, factorize the left-hand side $(x-1)(x-4) \geq 0$, use the fact that $AB \geq 0$ if $A \geq 0 \wedge B \geq 0$ or $A \leq 0 \wedge B \leq 0$. The solution is $x \leq 1$ or $x \geq 4$. Correspondingly, this is the part of the x -axis where $x^2 - 5x + 4$ is above the x -axis.)
3. $f(x) = \frac{1}{\sqrt{x^2 - 5x + 4}}$

Linear functions

$$f(x) = ax + b.$$

Exercises.

1. Find a linear function whose graph passes through $(2, -3)$ and $(0, 4)$.
2. Find a linear function whose graph passes through $(-2, 2)$ and $(2, 5)$.
3. Determine the x - and y -intercepts of

$$y = 3x - 6, \quad y = \frac{1}{2}x + 5, \quad y = x^2 - 3x + 2, \quad y = x^3 - x.$$

4. Find a linear function with slope $-\frac{1}{2}$ whose graph passes through $(5, -2)$.
5. A manufacturer's total cost consists of a fixed overhead of \$5,000 plus production costs of \$60 per unit. Write the total cost as a function of the number of units.

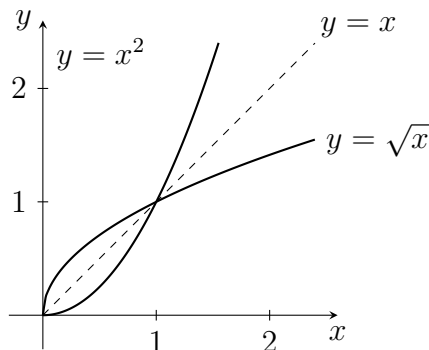
Exercises: determine the limit

1. $\lim_{x \rightarrow 2} (3x^2 - 5x + 2)$
2. $\lim_{x \rightarrow 3} (x - 1)^2(x + 1)$
3. $\lim_{x \rightarrow \frac{1}{3}} \frac{x + 1}{x + 2}$
4. $\lim_{x \rightarrow 5} \frac{x + 3}{5 - x}$
5. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$
6. $\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$
7. $\lim_{x \rightarrow 0} \frac{x(x^2 - 1)}{x^2}$
8. $\lim_{x \rightarrow -2} \frac{x^2 - x - 6}{x^2 + 3x + 2}$
9. $\lim_{x \rightarrow 2^-} \frac{x^2 + 4}{x - 2}$
10. $\lim_{x \rightarrow 1} \frac{x - \sqrt{x}}{\sqrt{x} - 1}$
11. $\lim_{x \rightarrow \infty} \frac{x^3 + 2x + 1}{10x - 1}$

12. $\lim_{x \rightarrow -\infty} \frac{x^3 + 5x^2}{-x^2 + 3}$

Power function

For $x \geq 0$, the equation $\sqrt{x} = y$ is equivalent to $x = y^2$. Therefore the graph of $f(x) = \sqrt{x}$ is obtained by reflecting the graph of $g(x) = x^2$ along the line $y = x$.



The graph of $f(x) = \sqrt{x}$ is connected, so $\sqrt{x}$ is continuous.

Similarly, we can define

$$\sqrt[3]{x} = x^{\frac{1}{3}} \quad (\text{cubic root}),$$

and more generally the power function x^a for real a . They are continuous on $[0, \infty)$.

2026.09.24: Exponential functions

Powers with a rational exponent

Let us recall that, for a number x ,

$$x^1 = x, \quad x^2 = x \cdot x, \quad x^3 = x \cdot x \cdot x, \quad \dots$$

and hence, for example,

$$x^2 \cdot x^3 = x^5, \quad (x^2)^3 = x^6,$$

and more generally $x^a x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$ for positive integers a and b .

Moreover, for $x > 0$ we saw that there is the square root $\sqrt{x}$, with $(\sqrt{x})^2 = x$. We write

$$\sqrt{x} = x^{\frac{1}{2}}$$

because $x = x^1 = (x^{\frac{1}{2}})^2$. Similarly, the n -th root is denoted by $x^{\frac{1}{n}}$.

In general, for a rational number $\frac{p}{q}$ (such as $\frac{1}{2}$, $\frac{5}{3}$), we write

$$x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = (x^p)^{\frac{1}{q}}.$$

For a negative exponent $-\frac{p}{q}$ we define, still for $x > 0$,

$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}}, \quad \text{and} \quad x^0 = 1.$$

Examples.

$$4^{\frac{1}{2}} = \sqrt{4} = 2, \quad 4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{\left(4^{\frac{1}{2}}\right)^3} = \frac{1}{2^3} = \frac{1}{8}, \quad 27^{-\frac{2}{3}} = \frac{1}{9}.$$

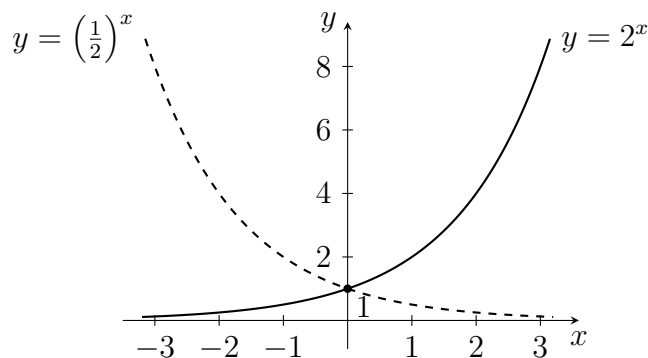
The exponential function

Now let $a > 0$ be constant. We can consider the function

$$f(x) = a^x,$$

first defined for rational numbers x , but we can define a^x for all real numbers x by continuity.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



Exponential rules

Let $a > 0$, $b > 0$ and $x, y \in \mathbb{R}$ (real numbers). Then

- if $b \neq 1$, then $b^x = b^y$ if and only if $x = y$;
- $b^x b^y = b^{x+y}$;
- $\frac{b^x}{b^y} = b^{x-y}$;

- $(b^x)^y = b^{xy}$;
- $(ab)^x = a^x b^x$;
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

Examples.

$$3^2 \cdot 3^3 = 3^5 = 243, \quad (2^3)^2 = 2^6 = 64, \quad \frac{2^3}{2^5} = 2^{-2} = \frac{1}{4}.$$

Properties of exponential functions

Let $b > 0$, $b \neq 1$.

- $f(x) = b^x$ is continuous, and $b^x > 0$;
- $b^0 = 1$;
- if $b > 1$, then b^x is increasing: if $x < y$, then $b^x < b^y$;
- if $0 < b < 1$, then b^x is decreasing: if $x < y$, then $b^x > b^y$.

Moreover, for $b > 1$,

$$\lim_{x \rightarrow \infty} b^x = \infty, \quad \lim_{x \rightarrow -\infty} b^x = 0.$$

Examples from the real world

Population

Let $P(x)$ be the population at the year x , and assume

$$P(2026) = 8.3 \text{ (billion)},$$

with a growth rate of 0.84% (assume it is constant). Then

$$P(2027) = 8.3 \cdot 1.0084, \quad P(2028) = 8.3 \cdot 1.0084^2,$$

and in general

$$P(x) = 8.3 \cdot 1.0084^{x-2026}.$$

Radioactivity

Caesium-137 is radioactive. Each atom of Cs-137 decays into Ba-137 by emitting an electron, with some probability. If there are X atoms, after 30 years half of them have decayed. So

$$f(t) = X \cdot \left(\frac{1}{2}\right)^{\frac{t}{30}}$$

is the number of Cs-137 atoms after t years.

Interest on an investment

If you have an amount of money P in your bank account for investment, with an interest rate r per year, then next year you have

$$B = P(1 + r).$$

After t years,

$$B(t) = P(1 + r)^t.$$

Instead of annual interest, let us see what happens if we compound the interest monthly,

$$P \left(1 + \frac{r}{12}\right)^{12t},$$

or daily,

$$P \left(1 + \frac{r}{365}\right)^{365t}.$$

Napier's number

What is the limit

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n?$$

n	10	100	1000
$\left(1 + \frac{1}{n}\right)^n$	2.5937	2.7048	2.7169

The limit is

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.71828 \dots,$$

called Napier's number.

Using Napier's number, we have

$$\lim_{n \rightarrow \infty} P \left(1 + \frac{r}{n}\right)^{nt} = Pe^{rt}.$$

Exponential vs polynomial

Recall that

$$(a + b)^2 = a^2 + 2ab + b^2, \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Expanding $(1 + 1)^n$ in the same way,

$$2^n = (1 + 1)^n = 1 + n + \frac{n(n-1)}{2} + \dots,$$

so, for $n \geq 2$,

$$\frac{n}{2^n} = \frac{n}{1 + n + \frac{n(n-1)}{2} + \dots} < \frac{n}{\frac{n(n-1)}{2}} = \frac{2}{n-1},$$

and hence

$$\lim_{n \rightarrow \infty} \frac{n}{2^n} = 0.$$

In general, for $m > 0$ and $a > 1$,

$$\lim_{x \rightarrow \infty} \frac{x^m}{a^x} = 0.$$

Exercises

1. Simplify $27^{\frac{5}{3}}$ and $(3^2)^{\frac{5}{2}}$.
2. Find x such that $4^{2x-1} = 16$.
3. Find x such that $3^x \cdot 2^{2x} = 144$.

2026.09.28: Logarithm

Suppose you invest 1000 euro with an annual interest rate of 8%. After one year you will have $1000 \cdot 1.08 = 1080$. How many years must pass before your investment doubles? That is, we look for x such that

$$1000 \cdot 1.08^x = 2000.$$

What is x ?

Definition

For $b > 0$, $b \neq 1$, and $x > 0$, we define $\log_b x$ to be the number such that

$$b^{\log_b x} = x.$$

That is,

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

Examples.

$$\begin{aligned} \log_{10} 1000 &= 3, & \log_2 32 &= 5, & \log_5 \frac{1}{125} &= -3, \\ \log_4 \frac{1}{2} &= -\frac{1}{2}, & \log_{64} 16 &= \frac{2}{3}. \end{aligned}$$

Logarithmic rules

Let $b > 0$, $b \neq 1$, and let $u, v > 0$. Then

- $\log_b 1 = 0$ and $\log_b b = 1$;
- $\log_b u = \log_b v$ if and only if $u = v$ (because $b^x = b^y$ if and only if $x = y$);
- $\log_b(uv) = \log_b u + \log_b v$ (because $b^x b^y = b^{x+y}$);
- $\log_b(u^r) = r \log_b u$;
- $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$.

For example,

$$b^{\log_b(uv)} = uv = b^{\log_b u} \cdot b^{\log_b v} = b^{\log_b u + \log_b v},$$

$$b^{\log_b(u^r)} = u^r = \left(b^{\log_b u}\right)^r = b^{r \log_b u}.$$

Examples. Write the following in terms of $\log_{10} 3 = 0.477\dots$ and $\log_{10} 2 = 0.301\dots$:

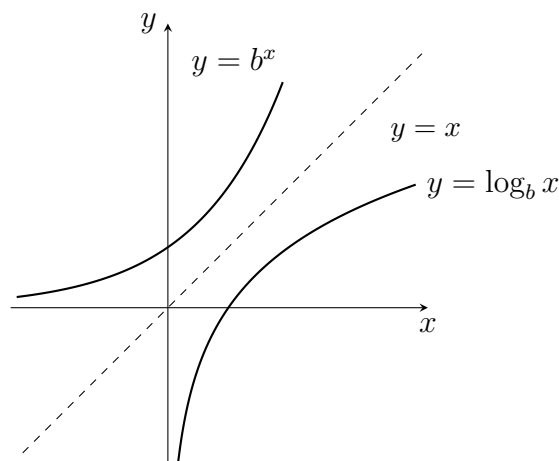
$$\log_{10} \frac{10}{3} = \log_{10} 10 - \log_{10} 3 = 1 - \log_{10} 3,$$

$$\log_{10} 8 = \log_{10} (2^3) = 3 \log_{10} 2,$$

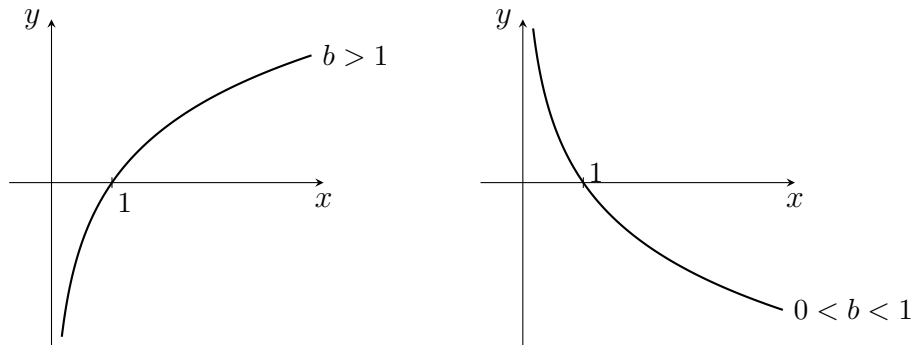
$$\log_{10} 36 = \log_{10} (2^2 \cdot 3^2) = 2 \log_{10} 2 + 2 \log_{10} 3.$$

Graph of the logarithmic function

Recall that $y = \log_b x$ if and only if $b^y = x$. For a given $x > 0$ there is such a y , by the intermediate value theorem. The graph of $y = \log_b x$ is obtained by reflecting the graph of $y = b^x$ along the line $y = x$.



The case $b > 1$ is drawn above. For $0 < b < 1$ the logarithm is decreasing:



Properties of a logarithmic function

Let $b > 0$, $b \neq 1$. The function $f(x) = \log_b x$, defined for $x > 0$, satisfies the following.

- It is continuous for all $x > 0$.
- If $b > 1$, then $\lim_{x \rightarrow 0^+} \log_b x = -\infty$, it is increasing, and $\lim_{x \rightarrow \infty} \log_b x = \infty$.
- If $0 < b < 1$, then $\lim_{x \rightarrow 0^+} \log_b x = \infty$, it is decreasing, and $\lim_{x \rightarrow \infty} \log_b x = -\infty$.
- The x -intercept is $(1, 0)$; there is no y -intercept.

When $b = e = 2.71828\dots$, the logarithm $\log_e x = \ln x$ is called the *natural logarithm*.

Inversion and conversion formulas

$$b^{\log_b x} = x \quad (\text{the definition}), \quad \log_b (b^x) = x.$$

Conversion formula.

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Indeed,

$$a^{\log_a x} = x = b^{\log_b x} = (a^{\log_a b})^{\log_b x} = a^{\log_a b \cdot \log_b x},$$

so $\log_a x = \log_a b \cdot \log_b x$.

Examples from the real world

Interest on an investment

If you invest 1000 euro with an 8% annual interest rate, after x years you will have

$$f(x) = 1000 \cdot (1.08)^x.$$

When will you have 2000 euro?

$$2000 = 1000 \cdot (1.08)^x \implies 2 = 1.08^x \implies x = \log_{1.08} 2 = \frac{\ln 2}{\ln 1.08} = 9.006\dots$$

that is, about 9 years.

Carbon dating

In the air there are the isotopes C-14, which is radioactive, and C-12, which is stable. In a living plant their ratio is a constant R_0 , but when the plant dies the ratio starts to decay exponentially, as

$$R_0 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5730}},$$

where the half-life of C-14 is 5730 years. This is the same form as the Cs-137 example above; note that

$$\left(\frac{1}{2}\right)^{\frac{t}{5730}} = 2^{-\frac{t}{5730}}.$$

If the ratio of C-14 to C-12 in a fossil is $\frac{R_0}{5}$, then that plant died t years ago, where

$$\frac{R_0}{5} = R_0 \cdot 2^{-\frac{t}{5730}}, \quad \text{hence} \quad -\log_2 5 = -\frac{t}{5730},$$

and therefore

$$t = 5730 \cdot \log_2 5 = 5730 \cdot \frac{\ln 5}{\ln 2} = 13304.6 \dots,$$

that is, about 13 300 years.

Exercise

If the population grows as

$$P(x) = 8.3 \cdot 1.0084^{x-2026}$$

(the model of the lesson of 09.24), when will it be 10 billion?

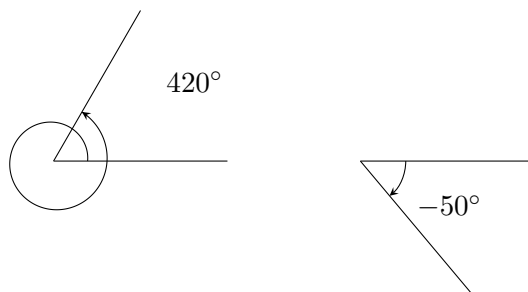
2026.09.29: Trigonometric functions, various limits

Angles and radians

Consider angles. Usually we measure them in degrees, from 0° to 360° (one rotation). We can also consider more than 360° (more than one rotation), or negative angles.

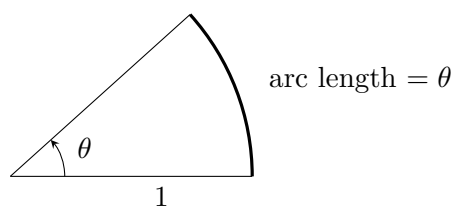


More than one rotation, and a negative angle:



We use the *radian* to measure angles: it is the length of the arc that the angle cuts out on the circle of radius 1. The conversion is

$$\frac{\text{degrees}}{180} = \frac{\text{radians}}{\pi},$$

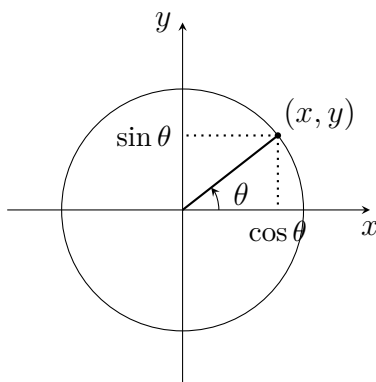


so that

$$360^\circ = 2\pi, \quad 60^\circ = \frac{\pi}{3}, \quad 45^\circ = \frac{\pi}{4}, \quad \pi = 3.14\dots$$

Sine and cosine

For an angle θ measured in radians, consider the point of the circle of radius 1 obtained from $(1, 0)$ by rotating counterclockwise by θ (clockwise, if $\theta < 0$). We define $x = \cos \theta$ and $y = \sin \theta$ to be the coordinates of that point.



Examples.

$$\begin{aligned}\cos 0 &= 1, & \cos \frac{\pi}{2} &= 0, & \cos \pi &= -1, \\ \sin 0 &= 0, & \sin \frac{\pi}{2} &= 1, & \sin \pi &= 0.\end{aligned}$$

As $360^\circ = 2\pi$ is one rotation, $\cos \theta$ and $\sin \theta$ are periodic:

$$\cos(\theta + 2\pi) = \cos \theta, \quad \sin(\theta + 2\pi) = \sin \theta.$$

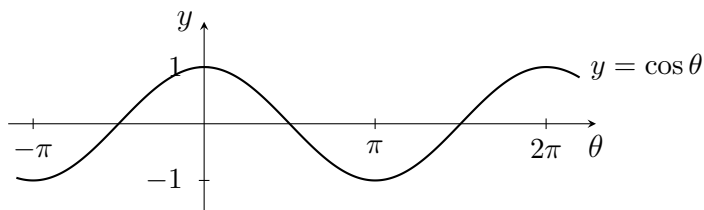
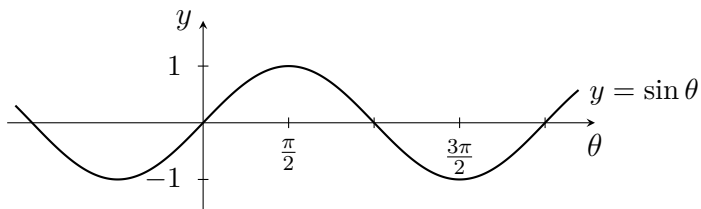
Other properties:

$$\cos(-\theta) = \cos \theta, \quad \sin(-\theta) = -\sin \theta,$$

and

$$(\cos \theta)^2 + (\sin \theta)^2 = 1,$$

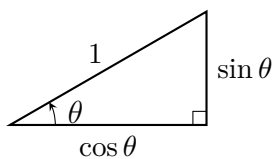
by the Pythagorean theorem.



Comparing the two graphs, the cosine is the sine shifted to the left by $\frac{\pi}{2}$:

$$\cos \theta = \sin \left(\theta + \frac{\pi}{2} \right).$$

Some values



$$\begin{aligned}\cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2}, & \cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}}, & \cos \frac{\pi}{3} &= \frac{1}{2}, \\ \sin \frac{\pi}{6} &= \frac{1}{2}, & \sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}}, & \sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Sometimes one uses

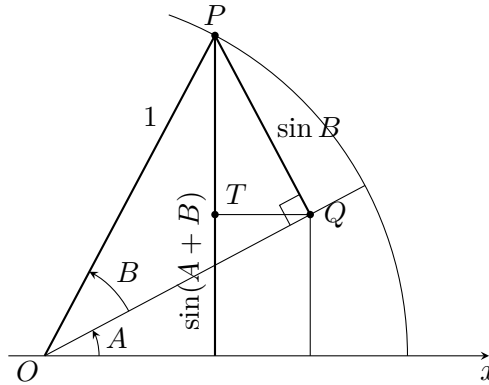
$$\tan \theta := \frac{\sin \theta}{\cos \theta}.$$

Some formulas:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

The formula for $\sin(A + B)$ can be read off the following picture, where $OP = 1$, the angle POQ is B and the angle QOx is A , so that $OQ = \cos B$ and $QP = \sin B$. The height of P is $\sin(A + B)$, and it is the sum of the height of Q , which is $\sin A \cos B$, and of the remaining piece TP , which is $\cos A \sin B$.



Various limits

In general, if $\lim_{x \rightarrow c} f(x) = L$ and g is continuous at L , then

$$\lim_{x \rightarrow c} g(f(x)) = g(L).$$

The continuity of g cannot be dropped: if $f(x) = 0$ for all x and $g(y) = 1$ for $y \neq 0$ while $g(0) = 0$, then $\lim_{y \rightarrow 0} g(y) = 1$, but $g(f(x)) = 0$ for all x .

First, we have

$$\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0^+} \ln \left((1+x)^{\frac{1}{x}} \right) = \lim_{y \rightarrow \infty} \ln \left(\left(1 + \frac{1}{y} \right)^y \right) = \ln e = 1.$$

Actually $\lim_{x \rightarrow 0^-} \frac{\ln(1+x)}{x} = 1$ as well, so

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1.$$

Next, putting $e^x - 1 = y$, that is $e^x = 1 + y$, that is $x = \ln(1 + y)$, we get

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{y \rightarrow 0} \frac{y}{\ln(1+y)} = 1.$$

Recall from the lesson of 09.24 that $\lim_{y \rightarrow \infty} \frac{y}{e^y} = 0$. With $x = e^{-y}$, where $y \rightarrow +\infty$,

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{y \rightarrow +\infty} e^{-y} \cdot \ln(e^{-y}) = \lim_{y \rightarrow \infty} \frac{-y}{e^y} = 0.$$

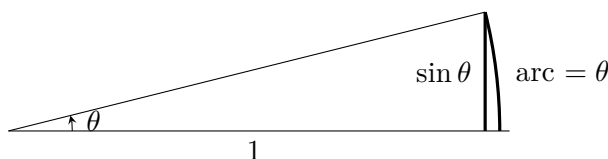
With $x = e^y$,

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{y \rightarrow \infty} \frac{\ln e^y}{e^y} = \lim_{y \rightarrow \infty} \frac{y}{e^y} = 0.$$

Finally,

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Indeed, on the circle of radius 1 the angle x is the length of the arc, while $\sin x$ is the length of the vertical segment below its endpoint. When x is small, the two are almost the same.



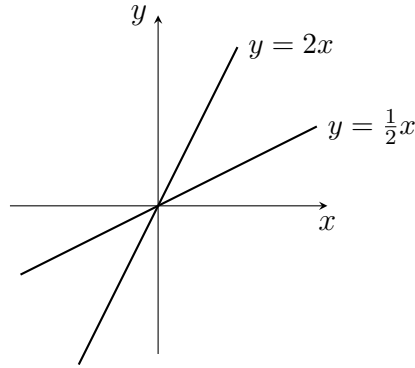
Exercises.

$$\begin{aligned} \lim_{x \rightarrow 0} \ln(1+x^2), \quad & \lim_{x \rightarrow \pi} \sin(\pi - x), \quad & \lim_{x \rightarrow 1} e^{x^2-1}, \\ \lim_{x \rightarrow 0} \frac{\sin 3x}{x}, \quad & \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}, \quad & \lim_{x \rightarrow 0} \frac{\ln(1+5x)}{x}. \end{aligned}$$

2026.09.30: Derivative

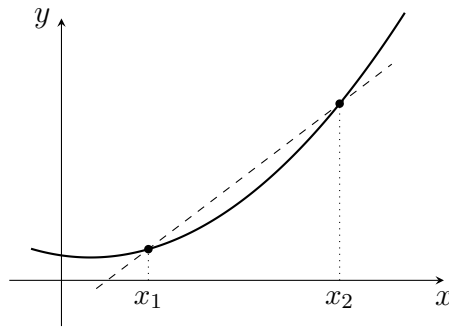
The slope of a segment

Recall the linear functions $f(x) = ax + b$. Here a is the slope of the graph: as x increases by 1, f increases by a .



For a general function $f(x)$, we can take two points on the graph and calculate the slope between them. If $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are such points, then the slope of the segment joining them is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$



Example. Assume that a car is moving on a straight road, and that at time x it is $f(x)$ kilometers away. The average velocity between time x_1 and time x_2 is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

What is the velocity at a single moment?

The derivative

The *derivative* of a function f at x_0 is the limit

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

if it exists. If this limit exists, we denote it by

$$f'(x_0), \quad \text{or} \quad \left. \frac{df}{dx} \right|_{x_0}.$$

In the example above, the velocity of the car at the single moment x_0 is $f'(x_0)$.

Example. For $f(x) = 2x + 1$ and $x_0 \in \mathbb{R}$,

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{(2(x_0 + h) + 1) - (2x_0 + 1)}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = \lim_{h \rightarrow 0} 2 = 2.$$

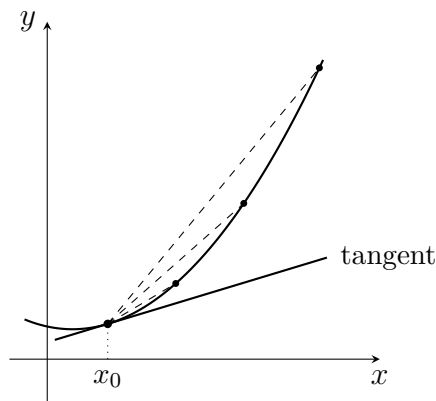
Example. For $f(x) = x^2$ and $x_0 \in \mathbb{R}$,

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{(x_0 + h)^2 - x_0^2}{h} = \lim_{h \rightarrow 0} \frac{x_0^2 + 2x_0h + h^2 - x_0^2}{h} = \lim_{h \rightarrow 0} (2x_0 + h) = 2x_0.$$

Recall that

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}$$

was the slope of the segment between $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$. As h becomes smaller, these segments (dashed in the picture) approach a limit position: the line *tangent* to the graph at $(x_0, f(x_0))$, whose slope is $f'(x_0)$.



If $f'(x)$ exists for all x in the domain, then $f'(x)$ is a new function, called the *derivative* of f .

Rules of the derivative

Let f and g have derivatives at x_0 . Then they are continuous at x_0 : indeed

$$\lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0)) = \lim_{h \rightarrow 0} h \cdot \frac{f(x_0 + h) - f(x_0)}{h} = 0 \cdot f'(x_0) = 0.$$

By the rules of limits,

$$\begin{aligned} (f + g)'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) + g(x_0 + h) - f(x_0) - g(x_0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} + \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} \\ &= f'(x_0) + g'(x_0). \end{aligned}$$

For the product,

$$\begin{aligned}
(f \cdot g)'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0 + h)}{h} + \lim_{h \rightarrow 0} \frac{f(x_0)g(x_0 + h) - f(x_0)g(x_0)}{h} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0).
\end{aligned}$$

For the quotient, assuming that $g(x_0) \neq 0$,

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{f(x_0 + h)}{g(x_0 + h)} - \frac{f(x_0)}{g(x_0)} \right) = \lim_{h \rightarrow 0} \frac{f(x_0 + h)g(x_0) - f(x_0)g(x_0 + h)}{h g(x_0 + h)g(x_0)} \\
&= \lim_{h \rightarrow 0} \frac{f(x_0 + h)g(x_0) - f(x_0)g(x_0)}{h g(x_0)g(x_0 + h)} + \lim_{h \rightarrow 0} \frac{f(x_0)g(x_0) - f(x_0)g(x_0 + h)}{h g(x_0 + h)g(x_0)} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.
\end{aligned}$$

Examples. Writing $x^3 = x^2 \cdot x$,

$$(x^3)' = (x^2)' \cdot x + x^2 \cdot (x)' = 2x \cdot x + x^2 \cdot 1 = 3x^2,$$

and writing $x^4 = x^3 \cdot x$,

$$(x^4)' = (x^3)' \cdot x + x^3 \cdot (x)' = 3x^2 \cdot x + x^3 \cdot 1 = 4x^3.$$

In general,

$$(x^n)' = nx^{n-1}.$$

Further examples: for $f(x) = A$ constant,

$$f'(x) = \lim_{h \rightarrow 0} \frac{A - A}{h} = 0,$$

and hence for $f(x) = Ag(x)$,

$$f'(x) = A'g(x) + Ag'(x) = Ag'(x).$$

Therefore

$$(5x^2 - 3x + 2)' = 5 \cdot 2x - 3 \cdot 1 + 0 = 10x - 3.$$

For $f(x) = \frac{1}{x}$,

$$f'(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2},$$

and for $f(x) = \frac{x^2 - 2}{x + 1}$,

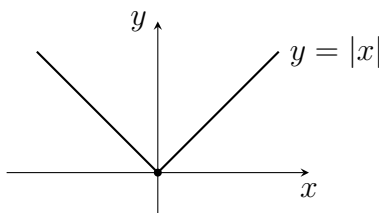
$$f'(x) = \frac{2x(x + 1) - (x^2 - 2) \cdot 1}{(x + 1)^2} = \frac{2x^2 + 2x - x^2 + 2}{(x + 1)^2} = \frac{x^2 + 2x + 2}{(x + 1)^2}.$$

Not all functions are differentiable

For $f(x) = |x|$ at 0,

$$\lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1, \quad \lim_{h \rightarrow 0^-} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1,$$

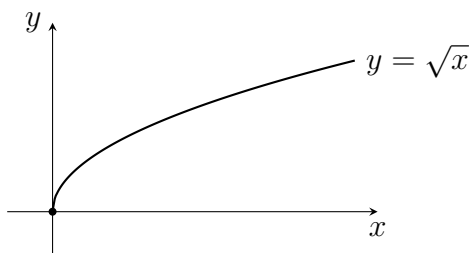
so the two one-sided limits differ and $f'(0)$ does not exist.



For $f(x) = \sqrt{x}$ at 0,

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - 0}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty,$$

so again $f'(0)$ does not exist: the tangent line is vertical.



2026.10.01: Derivatives and the chain rule

We defined the derivative of a function f at x_0 by

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Example. Let $f(x) = e^x$ and $x_0 \in \mathbb{R}$. We know that $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$, so

$$\lim_{h \rightarrow 0} \frac{e^{x_0+h} - e^{x_0}}{h} = \lim_{h \rightarrow 0} \frac{e^{x_0} (e^h - 1)}{h} = e^{x_0}.$$

That is,

$$(e^x)' = e^x.$$

Example. Let $f(x) = \ln x$ and $x_0 > 0$. We know that $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$, so, writing $h' = \frac{h}{x_0}$,

$$\lim_{h \rightarrow 0} \frac{\ln(x_0 + h) - \ln x_0}{h} = \lim_{h \rightarrow 0} \frac{\ln\left(\frac{x_0+h}{x_0}\right)}{h} = \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{h}{x_0}\right)}{\frac{h}{x_0} \cdot x_0} = \lim_{h' \rightarrow 0} \frac{\ln(1 + h')}{h' \cdot x_0} = \frac{1}{x_0}.$$

That is,

$$(\ln x)' = \frac{1}{x}.$$

Example. Let $f(x) = \sin x$. Recall that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B) = \sin A \cos B - \cos A \sin B,$$

and hence

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B.$$

Taking $A = x_0 + \frac{h}{2}$ and $B = \frac{h}{2}$, so that $x_0 + h = A + B$ and $x_0 = A - B$,

$$\lim_{h \rightarrow 0} \frac{\sin(x_0 + h) - \sin x_0}{h} = \lim_{h \rightarrow 0} \frac{2 \cos\left(x_0 + \frac{h}{2}\right) \sin \frac{h}{2}}{h} = \lim_{h \rightarrow 0} \cos\left(x_0 + \frac{h}{2}\right) \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}} = \cos x_0,$$

where in the last step we used $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$ with $u = \frac{h}{2}$. That is,

$$(\sin x)' = \cos x.$$

Example. Let $f(x) = \cos x$. We have

$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B,$$

hence, in the same way,

$$\lim_{h \rightarrow 0} \frac{\cos(x_0 + h) - \cos x_0}{h} = \lim_{h \rightarrow 0} \frac{-2 \sin\left(x_0 + \frac{h}{2}\right) \sin \frac{h}{2}}{h} = -\sin x_0.$$

That is,

$$(\cos x)' = -\sin x.$$

The chain rule

Let f and g be differentiable, and recall the composed function $f(g(x))$. Writing $h' = g(x_0 + h) - g(x_0)$,

$$\lim_{h \rightarrow 0} \frac{f(g(x_0 + h)) - f(g(x_0))}{h} = \lim_{h \rightarrow 0} \frac{f(g(x_0 + h)) - f(g(x_0))}{g(x_0 + h) - g(x_0)} \cdot \frac{g(x_0 + h) - g(x_0)}{h}$$

$$\begin{aligned}
&= \lim_{h' \rightarrow 0} \frac{f(g(x_0) + h') - f(g(x_0))}{h'} \cdot \lim_{h \rightarrow 0} \frac{g(x_0 + h) - g(x_0)}{h} \\
&= f'(g(x_0)) g'(x_0).
\end{aligned}$$

That is,

$$(f(g(x)))' = f'(g(x)) g'(x).$$

Examples. With $f(x) = e^x$ and $g(x) = x^2$, so that $e^{x^2} = f(g(x))$,

$$(e^{x^2})' = e^{x^2} \cdot 2x.$$

With $f(x) = \sin x$ and $g(x) = e^x$, so that $\sin(e^x) = f(g(x))$,

$$(\sin(e^x))' = \cos(e^x) \cdot e^x.$$

Let $\alpha \in \mathbb{R}$ and $x > 0$. Then

$$x^\alpha = (e^{\ln x})^\alpha = e^{\alpha \ln x} = f(g(x)), \quad f(x) = e^x, \quad g(x) = \alpha \ln x,$$

so that

$$(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \cdot \frac{\alpha}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

We saw this already for $\alpha \in \mathbb{N}$. For example,

$$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

With $f(x) = \sqrt{x}$ and $g(x) = x - 1$,

$$(\sqrt{x-1})' = \frac{1}{2\sqrt{x-1}},$$

and with $f(x) = x^3$ and $g(x) = x^2 + 3x + 1$,

$$\left((x^2 + 3x + 1)^3\right)' = 3(x^2 + 3x + 1)^2 \cdot (2x + 3).$$

In the same way we can differentiate the exponential and the logarithm in any base. For $a > 0$ we write $a^x = e^{x \ln a}$, so

$$(a^x)' = e^{x \ln a} \cdot \ln a = a^x \ln a,$$

and, by the conversion formula $\log_b x = \frac{\ln x}{\ln b}$ of the lesson of 09.28 (here $b > 0$, $b \neq 1$),

$$(\log_b x)' = \frac{1}{\ln b} \cdot (\ln x)' = \frac{1}{x \ln b}.$$

For $a = e$ and $b = e$ we recover $(e^x)' = e^x$ and $(\ln x)' = \frac{1}{x}$.

Summary

$f(x)$	$f'(x)$	valid for
A (constant)	0	all x
x^α	$\alpha x^{\alpha-1}$	$x > 0$ (all x , if $\alpha \in \mathbb{N}$)
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$x > 0$
e^x	e^x	all x
a^x	$a^x \ln a$	all x (here $a > 0$)
$\ln x$	$\frac{1}{x}$	$x > 0$
$\log_b x$	$\frac{1}{x \ln b}$	$x > 0$ (here $b > 0$, $b \neq 1$)
$\sin x$	$\cos x$	all x
$\cos x$	$-\sin x$	all x

together with the rules

$$(f + g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

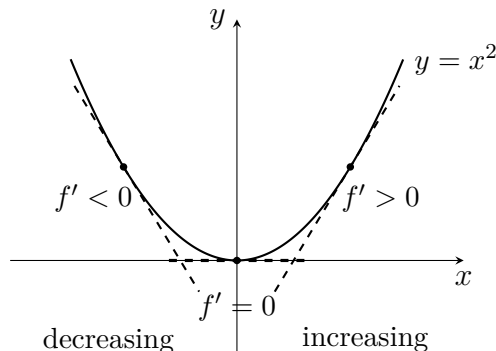
$$(f(g(x)))' = f'(g(x)) g'(x).$$

The derivative and the behaviour of the function

Recall that $f'(x)$ is the slope of the tangent line at $(x, f(x))$. If $f'(x) > 0$, the tangent points upwards; if $f'(x) < 0$, it points downwards.

If $f'(x) > 0$ on an interval, say (a, b) , then f is *increasing* there: for $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, it holds that $f(x_1) < f(x_2)$. If $f'(x) < 0$ on an interval (a, b) , then f is *decreasing* there.

Example. For $f(x) = x^2$ we have $f'(x) = 2x$. Hence $f'(x) > 0$ on $(0, \infty)$, where f is increasing, and $f'(x) < 0$ on $(-\infty, 0)$, where f is decreasing.



2026.10.05: Applications of the derivative

Increasing and decreasing functions

Let f be a function on a domain A and I be an interval in A . We write $x \in I$ to say that x is in I . We say that

- f is *increasing* in I if, for any pair $x_1, x_2 \in I$ with $x_1 < x_2$, it holds that $f(x_1) < f(x_2)$.
- f is *decreasing* in I if, for any pair $x_1, x_2 \in I$ with $x_1 < x_2$, it holds that $f(x_1) > f(x_2)$.

Example. $f(x) = x^2$ on $\mathbb{R}$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

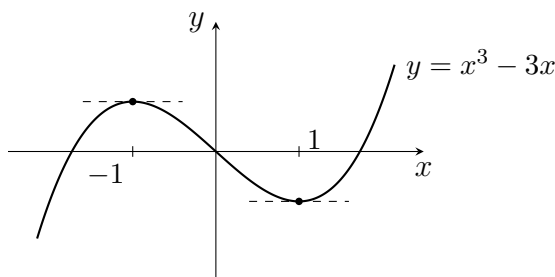
Assume now that A is an interval (a, b) , and recall that $f'(x)$ is the slope of the tangent line at $(x, f(x))$. Assume that f is differentiable at every $x \in (a, b)$. Then

- if $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b) ;
- if $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b) .

Example. For $f(x) = x^3 - 3x$ we have $f'(x) = 3x^2 - 3$, so

$$\begin{aligned} f'(x) > 0 &\iff 3(x^2 - 1) > 0 \iff x < -1 \text{ or } x > 1, \\ f'(x) < 0 &\iff 3(x^2 - 1) < 0 \iff -1 < x < 1. \end{aligned}$$

Hence f is increasing on $(-\infty, -1)$ and on $(1, \infty)$, and decreasing on $(-1, 1)$.



Relative extrema and critical points

We say that

- f has a *relative minimum* at x_0 if f is decreasing slightly before x_0 and increasing slightly after x_0 .
- f has a *relative maximum* at x_0 if f is increasing slightly before x_0 and decreasing slightly after x_0 .

In other words, a relative minimum is a valley and a relative maximum is a peak.

If x_0 is a relative minimum, then $f(x_0) \leq f(x)$ for x close to x_0 ; for a relative maximum, $f(x_0) \geq f(x)$ for x close to x_0 .

If f is differentiable at x_0 and has a relative minimum or maximum there, then the tangent line at $(x_0, f(x_0))$ is horizontal, so $f'(x_0) = 0$.

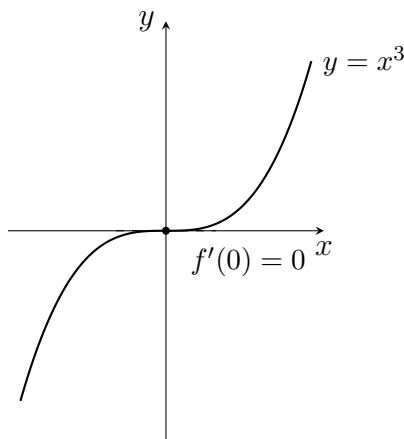
We say that x_0 is a *critical point* of f if $f'(x_0) = 0$. The discussion above says: if f is differentiable at x_0 and has a relative minimum or maximum there, then x_0 is a critical point. So, if f is differentiable, the critical points are the candidates for relative minima and maxima (*relative extrema*).

Example. For $f(x) = x^3 - 3x$ we have $f'(x) = 3x^2 - 3$. To find the critical points we set

$$f'(x_0) = 3x_0^2 - 3 = 0, \quad \text{so} \quad x_0 = \pm 1.$$

Saddle points

Not all critical points are relative extrema. For example, $f(x) = x^3$ has $f'(x) = 3x^2$ and $f'(0) = 0$, but $x^3 < 0$ if $x < 0$ and $x^3 > 0$ if $x > 0$, so 0 is neither a relative minimum nor a relative maximum.



If $f'(x_0) = 0$ but x_0 is neither a relative minimum nor a relative maximum, then x_0 is called a *saddle point* of f .

To decide whether a critical point x_0 is a relative extremum or a saddle point, we can look at the sign of the derivative:

- if $f'(x) < 0$ before x_0 and $f'(x) > 0$ after x_0 , then x_0 is a relative minimum;
- if $f'(x) > 0$ before x_0 and $f'(x) < 0$ after x_0 , then x_0 is a relative maximum.

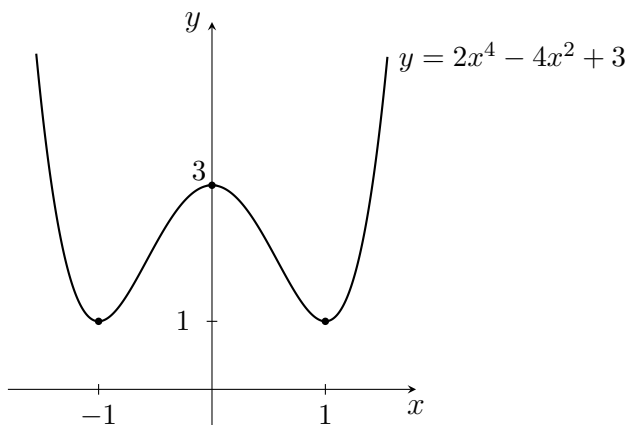
Example. Let $f(x) = 2x^4 - 4x^2 + 3$, so that $f'(x) = 8x^3 - 8x$. The critical points satisfy

$$f'(x_0) = 8x_0^3 - 8x_0 = 8x_0(x_0 - 1)(x_0 + 1) = 0, \quad \text{so} \quad x_0 = -1, 0, 1.$$

Moreover $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $-1 < x < 0$, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. We can summarise this in a table:

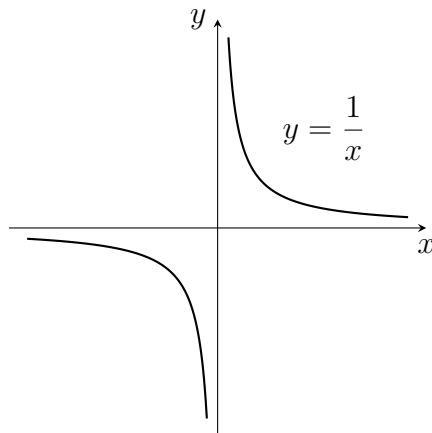
x	-1		0		1		
$f'(x)$	-	0	+	0	-	0	+
$f(x)$	$\searrow$	1	$\nearrow$	3	$\searrow$	1	$\nearrow$

Hence $x_0 = -1$ and $x_0 = 1$ are relative minima, and $x_0 = 0$ is a relative maximum.



Asymptotes

- If $\lim_{x \rightarrow c^+} |f(x)| = \infty$ or $\lim_{x \rightarrow c^-} |f(x)| = \infty$, that is, the graph of f goes up or down without bound as x approaches c from at least one side, we say that $x = c$ is a *vertical asymptote* of f . For $f(x) = \frac{1}{x}$, the line $x = 0$ is a vertical asymptote.
- If $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$, the graph of f approaches the horizontal line at height L , and we say that $y = L$ is a *horizontal asymptote*. For $f(x) = \frac{1}{x}$, the line $y = 0$ is a horizontal asymptote.



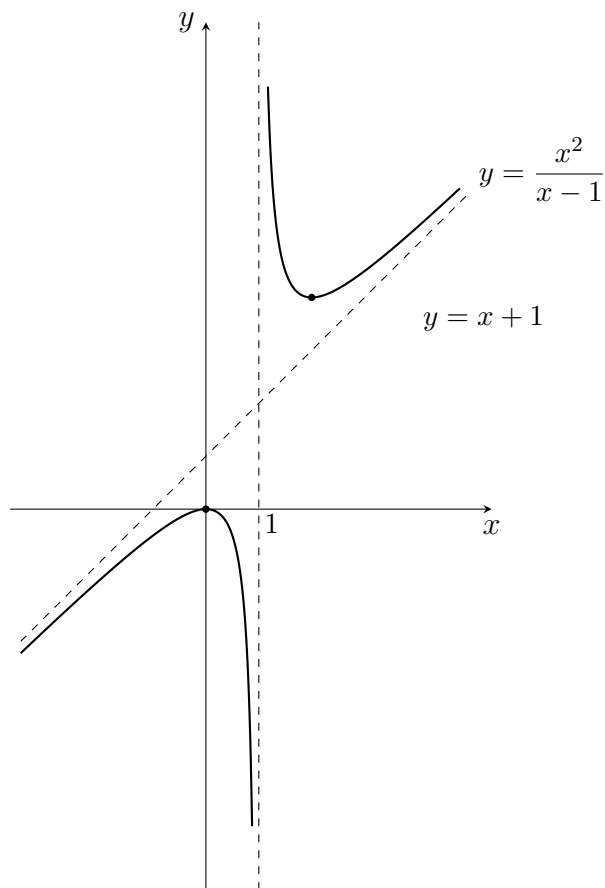
- It may happen that the graph of f approaches a line $y = ax + b$, called an *oblique asymptote*. We can compute

$$a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}, \quad b = \lim_{x \rightarrow \pm\infty} (f(x) - ax).$$

Example. For $f(x) = \frac{x^2}{x-1}$ we have

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x}{x-1} = 1, \quad \lim_{x \rightarrow \infty} (f(x) - x) = \lim_{x \rightarrow \infty} \frac{x}{x-1} = 1,$$

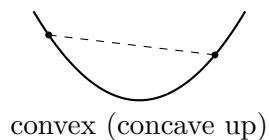
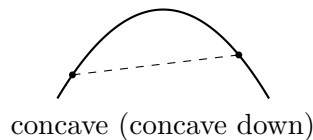
so $y = x + 1$ is an oblique asymptote.



2026.10.06: Higher derivatives

Convexity and concavity

Let (a, b) be contained in the domain of f . The graph of f is said to be *convex* on (a, b) if, for any pair $x_1 < x_2$ in (a, b) , the graph of f between x_1 and x_2 lies *below* the segment joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and *concave* if it lies *above* that segment.



Assume that f is differentiable on (a, b) . If the graph of f is convex, then the slope $f'(x)$ increases on (a, b) . So we study whether f' is increasing or decreasing.

The second derivative

Assume that $f'(x)$ exists for $x \in (a, b)$ and that f' is again differentiable. The derivative of f' is called the *second derivative* and is denoted by

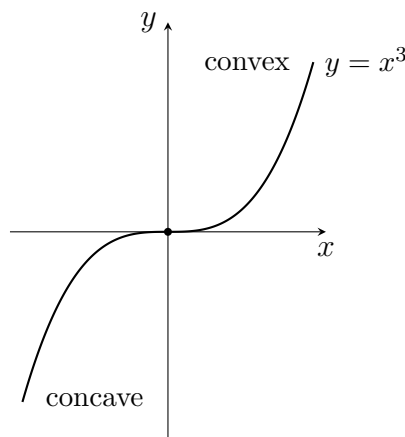
$$f''(x), \quad \text{or} \quad \frac{d^2 f}{dx^2}.$$

In the same way we can consider the third, fourth, ... derivatives.

Examples. For $f(x) = x^2$ we have $f'(x) = 2x$ and $f''(x) = 2 > 0$, so the graph of x^2 is convex.

For $f(x) = x^3$ we have $f'(x) = 3x^2$ and $f''(x) = 6x$, so

$$f''(x) > 0 \iff x > 0 \quad (\text{convex}), \quad f''(x) < 0 \iff x < 0 \quad (\text{concave}).$$



A point where the graph changes from being concave to convex (or vice versa) is called a *point of inflection*. For $f(x) = x^3$, the point $x = 0$ is a point of inflection. Since also $f'(0) = 0$, it is the saddle point of x^3 that we saw before: a point of inflection with a horizontal tangent line.

The second derivative and critical points

Assume that f'' exists on (a, b) and that $x_0 \in (a, b)$ is a critical point of f , so that $f'(x_0) = 0$.

If x_0 is a relative minimum, then f decreases slightly before x_0 and increases slightly after x_0 , so $f'(x) < 0$ slightly before x_0 and $f'(x) > 0$ slightly after x_0 : that is, f' must be increasing around x_0 . Similarly, if x_0 is a relative maximum, then f' must be decreasing around x_0 . Conversely, if $f''(x_0) > 0$, then f' is increasing around x_0 ; since $f'(x_0) = 0$, we get $f'(x) < 0$ slightly before x_0 and $f'(x) > 0$ slightly after x_0 , so x_0 is a relative minimum. The case $f''(x_0) < 0$ is similar. Hence

$$f''(x_0) > 0 \implies x_0 \text{ is a relative minimum,}$$

$$f''(x_0) < 0 \implies x_0 \text{ is a relative maximum.}$$

If $f''(x_0) = 0$, this test says nothing.

Example. Let $f(x) = x^3 - 3x$, so $f'(x) = 3x^2 - 3$. The critical points satisfy $f'(x_0) = 3(x_0^2 - 1) = 0$, so $x_0 = \pm 1$. As $f''(x) = 6x$,

$$f''(1) = 6 > 0, \quad f''(-1) = -6 < 0,$$

so $x_0 = 1$ is a relative minimum and $x_0 = -1$ is a relative maximum.

Curve sketching

Using all the tools above, we can sketch the graph of a function.

1. Find the domain of f , and the points of discontinuity and of non-differentiability.
2. Find the y -intercept and the x -intercepts.
3. Find the asymptotes (vertical, horizontal, oblique).
4. Compute $f'(x)$ and find the critical points, $f'(x_0) = 0$. Compute $f(x_0)$. Study the sign of $f'(x)$ to find where f is increasing and where it is decreasing.
5. Classify the critical points into relative minima, maxima and saddle points, using the sign of $f'(x)$, or $f''(x_0)$ when $f''(x_0) \neq 0$.
6. (optional) Study $f''(x)$ and the convexity and concavity.
7. Draw the graph.

Example. Let

$$f(x) = \frac{x}{(x+1)^2}.$$

1. Domain: $x \neq -1$. There f is continuous and differentiable.
2. y -intercept: $f(0) = 0$. x -intercept: $\frac{x}{(x+1)^2} = 0$ if and only if $x = 0$.
3. $\lim_{x \rightarrow -1} \frac{x}{(x+1)^2} = -\infty$, so $x = -1$ is a vertical asymptote. Moreover

$$\lim_{x \rightarrow \infty} \frac{x}{(x+1)^2} = 0, \quad \lim_{x \rightarrow -\infty} \frac{x}{(x+1)^2} = 0,$$

so $y = 0$ is a horizontal asymptote.

4. We compute

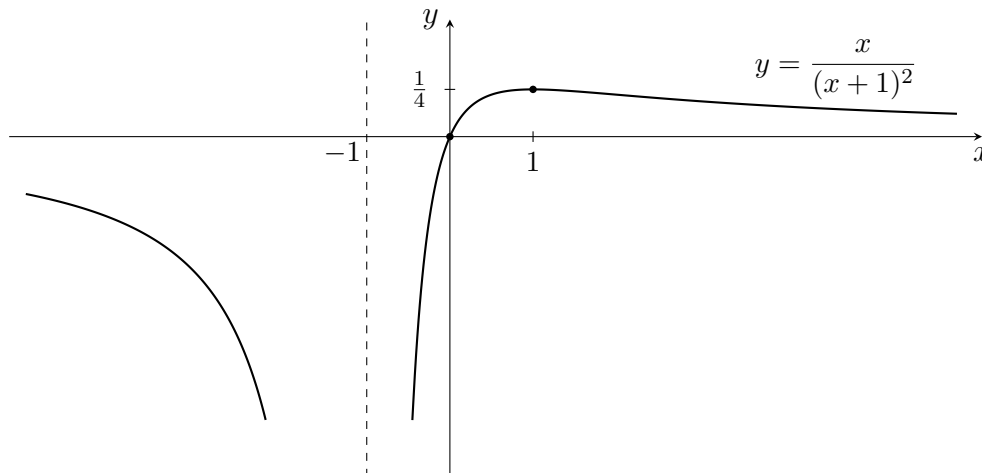
$$f'(x) = \frac{(x+1)^2 - x \cdot 2(x+1)}{(x+1)^4} = \frac{-x^2 + 1}{(x+1)^4}.$$

The critical points satisfy

$$f'(x_0) = \frac{-x_0^2 + 1}{(x_0 + 1)^4} = 0, \quad \text{so} \quad x_0 = \pm 1,$$

but -1 is not in the domain. Moreover $f(1) = \frac{1}{4}$.

5. (The second derivative would be long.) In $f'(x)$ the denominator is positive, so $f'(x)$ has the same sign as $-x^2 + 1$: $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ for $x > 1$. Hence f is decreasing on $(-\infty, -1)$, increasing on $(-1, 1)$ and decreasing on $(1, \infty)$, and $x_0 = 1$ is a relative maximum.



2026.10.07: Exercises, absolute extrema

A curve sketching exercise

Let

$$f(x) = \frac{3x^2}{x^2 + 2x - 15}.$$

1. Domain: we need

$$x^2 + 2x - 15 \neq 0 \iff (x + 5)(x - 3) \neq 0 \iff x \neq -5, 3.$$

2. y -intercept: $f(0) = 0$. x -intercept: $\frac{3x^2}{x^2 + 2x - 15} = 0$ if and only if $x = 0$.
 3. Asymptotes. There are vertical asymptotes at $x = -5$ and $x = 3$, as there the denominator tends to 0 while the numerator remains nonzero. For the horizontal ones,

$$\lim_{x \rightarrow \infty} \frac{3x^2}{x^2 + 2x - 15} = 3 = \lim_{x \rightarrow -\infty} \frac{3x^2}{x^2 + 2x - 15},$$

so $y = 3$ is a horizontal asymptote. There is no oblique asymptote.

4. We compute

$$f'(x) = \frac{6x(x^2 + 2x - 15) - 3x^2(2x + 2)}{(x^2 + 2x - 15)^2} = \frac{6x^2 - 90x}{(x^2 + 2x - 15)^2}.$$

The critical points satisfy

$$f'(x_0) = \frac{6x_0^2 - 90x_0}{(x_0^2 + 2x_0 - 15)^2} = 0 \iff 6x_0(x_0 - 15) = 0 \iff x_0 = 0, 15,$$

and

$$f(0) = 0, \quad f(15) = \frac{3 \cdot 15^2}{15^2 + 2 \cdot 15 - 15} = \frac{675}{240} = \frac{45}{16}.$$

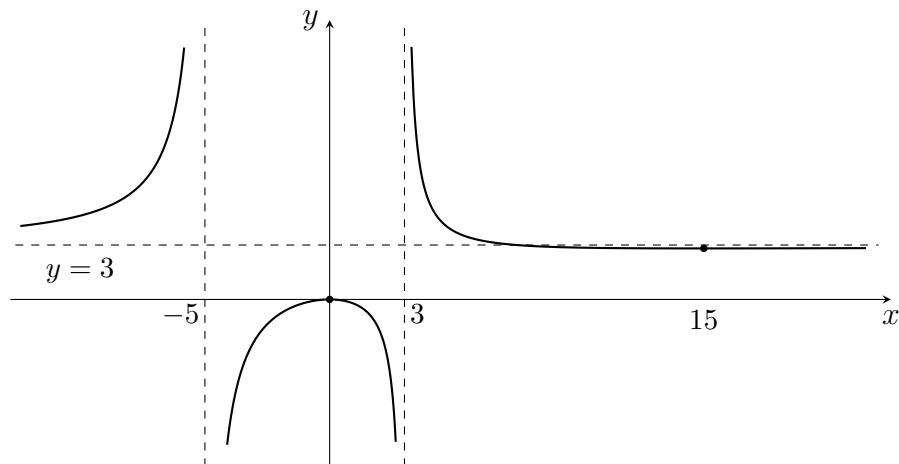
5. As the denominator is positive,

$$f'(x) > 0 \iff 6x^2 - 90x > 0 \iff x < 0 \ (x \neq -5) \text{ or } x > 15,$$

and $f'(x) < 0$ for $0 < x < 15$ (excluding $x = 3$).

x	-5			0			3			15		
$f'(x)$	+			+	0	-				-	0	+
$f(x)$	$\nearrow$			$\nearrow$	0	$\searrow$				$\searrow$	$\frac{45}{16}$	$\nearrow$

Hence $x_0 = 0$ is a relative maximum and $x_0 = 15$ is a relative minimum.



Example. Let $f(x) = 2x^4 - 4x^2 + 3$. Classify the critical points. We have $f'(x) = 8x^3 - 8x$, so

$$f'(x_0) = 8x_0^3 - 8x_0 = 0 \iff x_0 = -1, 0, 1,$$

and $f''(x) = 24x^2 - 8$, so

$$f''(-1) = 16 > 0, \quad f''(0) = -8 < 0, \quad f''(1) = 16 > 0.$$

Hence $x_0 = -1$ and $x_0 = 1$ are relative minima, and $x_0 = 0$ is a relative maximum. This agrees with what we found on 2026.10.05 using the sign of f' .

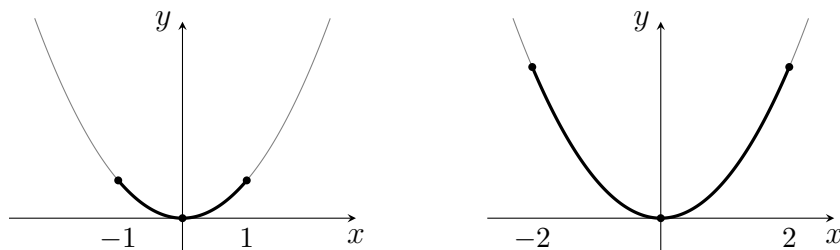
Absolute extrema

Let f be a function defined on a domain I . We say that f has an *absolute maximum* at $c \in I$ if $f(c) \geq f(x)$ for all $x \in I$, and an *absolute minimum* at c if $f(c) \leq f(x)$ for all $x \in I$.

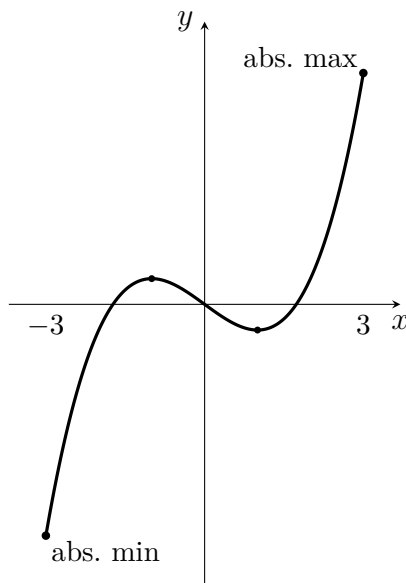
Instead, f has a *relative maximum* at $c \in I$ if $f(c) \geq f(x)$ for x close to c , and a *relative minimum* if $f(c) \leq f(x)$ for x close to c .

Relative maxima and minima can be absolute maxima and minima, but need not be. Absolute maxima and minima depend on the domain.

Examples. For $f(x) = x^2$ on $[-1, 1]$, the absolute maximum 1 is taken at -1 and at 1 . On $[-2, 2]$, the absolute maximum 4 is taken at -2 and at 2 . In both cases the absolute minimum 0 is taken at 0.



For $f(x) = x^3 - 3x$ on $[-3, 3]$, the points $x_0 = -1$ and $x_0 = 1$ are a relative maximum and a relative minimum, but the absolute maximum and the absolute minimum are taken at 3 and -3 , with $f(3) = 18$ and $f(-3) = -18$.



Application: marginal analysis

In economics, the derivative is used to analyse the change in a quantity that results from a one-unit (small) increase in production.

If $C(x)$ is the total cost of producing x units of a commodity, then the *marginal cost* at the production level x_0 is $C'(x_0)$. It approximates the cost $C(x_0 + 1) - C(x_0)$ of producing one more unit: since

$$\frac{C(x_0 + h) - C(x_0)}{h} \approx C'(x_0)$$

for small h , taking $h = 1$, which is small compared with x_0 , gives $C(x_0 + 1) - C(x_0) \approx C'(x_0)$. Analogously, if $R(x)$ is the revenue from producing x units, the *marginal revenue* is $R'(x_0)$. One wants to find the maximum of the profit $R(x) - C(x)$.

Example. A manufacturer produces x units of a commodity at the cost

$$C(x) = \frac{1}{8}x^2 + 3x + 98 \quad \text{euro},$$

and the price is $p(x) = \frac{1}{3}(75 - x)$ euro per unit, for $0 \leq x \leq 75$. The revenue is then $x p(x)$, so the profit is

$$P(x) = x p(x) - C(x).$$

What is the value of x which maximises the profit? We compute

$$P'(x) = \frac{1}{3}(75 - x) - \frac{x}{3} - \frac{1}{4}x - 3,$$

so $P'(x_0) = 0$ gives

$$25 - \frac{x_0}{3} - \frac{x_0}{3} - \frac{x_0}{4} - 3 = 0, \quad 22 = \frac{11}{12}x_0, \quad x_0 = \frac{12}{11} \cdot 22 = 24.$$