Mathematical Analysis I: Lecture 22

Lecturer: Yoh Tanimoto

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Start recording...

Annoucements

- Tutoring (by Mr. Lorenzo Panebianco): Tuesday 15:00-16:30 until 10th November. Then move to Tuesday morning.
- Today: Apostol Vol. 1, Chapter 4.5,10.

Derivative

For a function f defined on an open interval I and $x \in I$, we have defined the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, and we say that f is differentiable at x if this limit exists. Sometimes we denote this as $f'(x) = (Df)(x)$.
This is equivalent to write $Df(x) = f'(x) = \lim_{w \to x} \frac{f(w)-f(x)}{w-x}$.

Derivative

Let f, g be functions. We write this $x \mapsto f(x)$. We denote by $f + g$ the function that maps $x \mapsto f(x) + g(x)$. Similarly, $f \cdot g = fg$ is the function $x \mapsto f(x)g(x)$, $\frac{f}{g}$ is the function $x \mapsto \frac{f(x)}{g(x)}$, and the composition is $f \circ g$ that is given by $x \mapsto f(g(x))$.

Theorem

Let f, g be functions on open intervals. The following hold if f, g are differentiable at x (or f at $g(x)$ for the chain rule):

- *For $a, b \in \mathbb{R}$, $D(af + bg)(x) = aDf(x) + bDg(x)$ (linearity).*
- *$D(fg)(x) = Df(x)g(x) + f(x)Dg(x)$ (Leibniz rule).*
- *If $g(x) \neq 0$, then $D(\frac{f}{g})(x) = \frac{Df(x)g(x) - f(x)Dg(x)}{g(x)^2}$.*
- *$D(f \circ g) = Dg(x)Df(g(x))$ (the chain rule).*
- *If $Df(x) \neq 0$ and f is monotonically increasing or decreasing and continuous in $(x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$. Then f^{-1} is differentiable at $y = f(x)$ and $D(f^{-1}(y)) = \frac{1}{Df(x)}$.*

Proof.

Linearity is straightforward from the algebra of limits:

$$
\begin{aligned}
&\lim_{h\to 0} \frac{af(x+h)+bg(x+h)-af(x)-bg(x)}{h} \\
&= \lim_{h\to 0} a\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} b\frac{g(x+h)-g(x)}{h} \\
&= a\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + b\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= aDf(x)+bDg(x).
\end{aligned}
$$

Proof.

Note that $f(x+h)g(x+h) - f(x)g(x) = f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$, and g is continuous at x because it is differentiable there:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$
$$= Df(x)g(x) + f(x)Dg(x).$$

Proof.

As $g(x) \neq 0$, we have $\lim_{h\to 0} \frac{1}{g(x+h)} = \frac{1}{g(x)}$ and

$$\begin{aligned}
&\lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x) - f(x)(g(x+h) - g(x))}{g(x+h)g(x)h} \\
&= \frac{Df(x)g(x) - f(x)Dg(x)}{g(x)^2}.
\end{aligned}$$

Proof.

Note that $u(k) := \frac{f(g(x)+k)-f(g(x))}{k} - Df(g(x))$ tends to 0 as $k \to 0$. Let us also set $u(0) = 0$, then u is continuous around 0. We can write this as $f(g(x) + k) - f(g(x)) = k(Df(g(x)) + u(k))$, including $k = 0$.

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(g(x) + (g(x+h) - g(x))) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{(g(x+h) - g(x))(Df(g(x)) + u(g(x+h) - g(x)))}{h} \\
&= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot Df(g(x)) \\
&\qquad + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot u(g(x+h) - g(x)) \\
&= Dg(x)Df(g(x)),
\end{aligned}
$$

because $g(x + h)$ tends to $g(x)$, $u(k)$ is continuous and $u(0) = 0$.

Proof.

Let us assume that f is monotonically increasing and continuous in $(x-\epsilon, x+\epsilon)$. Then, with $y = f(x)$,

$$\begin{aligned}
\lim_{h\to 0} \frac{f^{-1}(y+h) - f^{-1}(y)}{h} &= \lim_{z\to y} \frac{f^{-1}(z) - f^{-1}(y)}{z-y} \\
&= \lim_{w\to x} \frac{f^{-1}(f(w)) - f^{-1}(f(x))}{f(w) - f(x)} \\
&= \lim_{w\to x} \frac{w-x}{f(w)-f(x)} = \frac{1}{Df(x)},
\end{aligned}$$

where in the second equality we used the change of variables $z = f(w)$. The case where f is monotonically decreasing is analogous. □

Derivative

Example

Let $f(x) = x^4 + 3x^2 - 34$.

Example

Let $f(x) = x^4 + 3x^2 - 34$. Then $Df(x) = 4x^3 + 6x$.

Derivative

Example

Let $f(x) = \frac{x^2+1}{x-2}$.

Derivative

Example

Let $f(x) = \frac{x^2+1}{x-2}$. Then, for $x \neq 2$, $Df(x) = \frac{2x(x-2)-(x^2+1)\cdot 1}{(x-2)^2} = \frac{x^2-4x-1}{(x-2)^2}$.

Derivative

Example

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Derivative

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Let us take the composition $\sin(x^2) = f(g(x))$.

Derivative

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By Leibniz rule, $D(x^2 \sin x) = 2x \sin x + x^2 \cos x$.
Let us take the composition $\sin(x^2) = f(g(x))$. By the chain rule, $D(\sin(x^2)) = D(x^2) \cdot (D \sin)(x^2) = 2x \cdot \cos(x^2)$. For $(\sin x)^2 = g(f(x))$,

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By Leibniz rule, $D(x^2 \sin x) = 2x \sin x + x^2 \cos x$.
Let us take the composition $\sin(x^2) = f(g(x))$. By the chain rule, $D(\sin(x^2)) = D(x^2) \cdot (D \sin)(x^2) = 2x \cdot \cos(x^2)$. For $(\sin x)^2 = g(f(x))$, $D((\sin x)^2) = D(\sin x) \cdot 2(\sin x) = 2 \sin x \cos x$.

Derivative

Example

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Derivative

Example

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Example

By the chain rule, $D(\exp(-x)) = D(-x) \cdot (D\exp)(-x) = -\exp(-x)$. By linearity, $D \sinh x = D(\frac{1}{2}(e^x - e^{-x})) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$. Analogously, $D \cosh x = \sinh x$.

Derivative

Example

For $a > 0$, it holds that $a^x = (e^{\log a})^x = e^{\log a \cdot x}$. Indeed, by the chain rule,

Derivative

Example

For $a > 0$, it holds that $a^x = (e^{\log a})^x = e^{\log a \cdot x}$. Indeed, by the chain rule,

$$\begin{aligned} D(a^x) &= D(\exp(\log a \cdot x)) = D(\log a \cdot x) \cdot (D\exp)(\log a \cdot x) \\ &= \log a \cdot \exp(\log a \cdot x) = \log a \cdot a^x. \end{aligned}$$

Derivative

Example

Let $a > 0$ and $f(x) = x^a$ for $x > 0$.

Derivative

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Derivative

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Let $a > 0$ and $f(x) = x^a$ for $x > 0$. $f(x) = \exp(\log x \cdot a)$, and by the chain rule,

$$\begin{aligned} Df(x) &= D(\log x \cdot a)D(\exp)(\log x \cdot a) \\ &= \frac{a}{x} \cdot \exp(\log x \cdot a) \\ &= \frac{a}{x} \cdot x^a = ax^{a-1}. \end{aligned}$$

For $a < 0$, we consider $f(x) = x^a = \frac{1}{x^a}$ and we obtain the same formula $f'(x) = ax^{a-1}$. For $a = 0$, because $x^a = 1$, we have $D(x^0) = D(1) = 0$.

Derivative

Example

$D \tan x = D(\frac{\sin x}{\cos x}) =$

Example

$$D \tan x = D\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Derivative

Example

$f(y) = \arctan y$. That is, $f(y) = g^{-1}(y)$, where $g(x) = \tan x$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Derivative

Example

$f(y) = \arctan y$. That is, $f(y) = g^{-1}(y)$, where $g(x) = \tan x$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By the formula for the inverse function, we have $Df(y) = \frac{1}{Dg(x)} = \cos^2 x$, where $y = g(x) = \tan x$. Therefore, $y^2 = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x}$, and $\cos^2 x = \frac{1}{1+y^2}$. By substituting this in the previous result, $D \arctan y = Df(y) = \frac{1}{1+y^2}$.

Derivative

- $f(x) = \tanh x$. $f'(x) = \frac{1}{\cosh^2 x}$.
- $f(x) = \arcsin x$ (the inverse function of $\sin x$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$). $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

Exercises

- Compute the derivative: $f(x) = x^2 - \cos(3x)$.
- Compute the derivative: $f(x) = \sqrt{x^2 + 1}$.
- Compute the derivative: $f(x) = \sin(\frac{x+2}{e^x})$.
- Compute the derivative: $f(x) = \sin(\cos(x^2))$.
- Compute the derivative: $f(x) = \log x$, using that $\log x$ is the inverse function of e^x.
- Compute the derivative: $f(x) = \sqrt{x}$ using that $\sqrt{2}$ is the inverse function of x^2.