BSc Engineering Sciences – A. Y. 2017/18

Written exam of the course Mathematical Analysis 2

June 20, 2018

Solve the following problems, motivating in detail the answers.

1. (6 points) Study the pointwise and uniform convergence, as $n \to \infty$, of the following sequence of functions:

$$f_n(x) := \sin(x + 2\pi\sqrt{n^2+1}), \qquad x \in \mathbb{R},\, n \in \mathbb{N}.$$

Solution. For any y, it holds that $\sin y = \sin(y - 2\pi n)$, therefore,

$$f_n(x) = \sin(x + 2\pi\sqrt{n^2+1} - 2\pi n).$$

Note also that

$$\sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n}.$$

Combining them, we obtain

$$\begin{aligned}
|f_n(x) - \sin x| &= |\sin(x + 2\pi(\sqrt{n^2+1}-n)) - \sin x| \\
&= \left|\sin\left(x + \frac{2\pi}{\sqrt{n^2+1}+n}\right) - \sin x\right| \\
&\le \max\left\{|\cos y| : x \le y \le x + \tfrac{2\pi}{\sqrt{n^2+1}+n}\right\} \cdot \frac{2\pi}{\sqrt{n^2+1}+n} \\
&\le \frac{2\pi}{n}.
\end{aligned}$$

where in the 3rd step we applied the mean value theorem to the function $\sin x$. The last expressioon $\frac{\pi}{n}$ tends to 0 and **this does not depend on** x, therefore, the sequence of functions $f_n(x)$ converges to $\sin x$ uniformly.

2.

(1) (4 points) Find all the stationary points of the following scalar field, defined on $\mathbb{R}^2$,

$$f(x,y) = e^y(x^2 - 2xy + 3)$$

and classify them into relative minima, maxima and saddle points.

(2) (2 points) Compute the derivative of the following function of $t \in \mathbb{R}$:

$$g(t) = \int_0^{\sin t} \cos e^s ds$$

Solution.
(1) We have

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= e^y(2x - 2y),\\ \frac{\partial f}{\partial y}(x,y) &= e^y(x^2 - 2xy + 3 - 2x).\end{aligned}$$

At any stationary point (x,y), one has $\nabla f(x,y) = (0,0)$. This happenes exactly when $e^y(2x-2y) = 0$ and $e^y(x^2-2xy+3-2x) = 0$. This is equivalent to $x = y$ and $x^2-2x^2+3-x = 0$. By solving these equations, the stationary points are $(x,y) = (1,1), (-3,-3)$.

To classify these points, we compute the Hessian matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial^2 x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{pmatrix}$:

$$H(x,y) = \begin{pmatrix} e^y \cdot 2 & e^y(2x - 2y - 2) \\ e^y(2x - 2y - 2) & e^y(x^2 - 2xy + 3 - 4x) \end{pmatrix}.$$

At the point $(x,y) = (1,1)$, we have $H(x,y) = \begin{pmatrix} 2e & -2e \\ -2e & -2e \end{pmatrix}$ and its determinant is $-8e^2 < 0$, threfore, it has a positive and a negative eigenvalues, and the point $(1,1)$ is a saddle.

At the point $(x,y) = (-3,-3)$, we have $H(x,y) = \begin{pmatrix} 2e^{-3} & -2e^{-3} \\ -2e^{-3} & 6e^{-3} \end{pmatrix}$ and its determinant is $8e^{-6} > 0$ and the trace is $8e^{-6} > 0$, threfore, it has two positive eigenvalues, and the point $(-3,-3)$ is a relative minimum.

(2) Let us put $F(x) = \int_0^x \cos e^s ds$ and $F'(x) = \cos e^x$, and $g(t) = F(\sin t)$. By the chain rule, $g'(t) = \cos t \cdot F'(\sin t) = \cos t \cdot \cos e^{\sin t}$.

Matriculation: ....

3.

(1) (3 points) Find the solution $f(x,y)$ of the partial differential equation

$$3\frac{\partial f}{\partial x} + 5\frac{\partial f}{\partial y} = 0$$

with the initial condition $f(x,0) = \cos(x^2)$.

(2) (3 points) Let $k > 0$ be a constant and $g(x,t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{kx^2}{t}\right)$. Determine the value of k for which $g(x,t), t > 0, x \in \mathbb{R}$ satisfies the following partial differential equation (1d heat equation):

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2}$$

Solution.

(1) The general solution of the differential equation is, with a continuously differentiable function $g(t)$, $f(x,y) = g(5x - 3y)$. With the given initial condition, it should hold that $f(x,0) = g(5x) = \cos(x^2)$, namely, $g(t) = \cos\frac{t^2}{25}$. Again by the general formula, $f(x,y) = g(5x-3y) = \cos\frac{(5x-3y)^2}{25}$.

(2) We have

$$\begin{aligned} \frac{\partial g}{\partial t} &= \left(-\frac{1}{2t^{\frac{3}{2}}} + \frac{kx^2}{t^{\frac{5}{2}}}\right)\exp\left(-\frac{kx^2}{t}\right), \\ \frac{\partial g}{\partial x} &= \frac{1}{\sqrt{t}}\left(-\frac{2kx}{t}\right)\exp\left(-\frac{kx^2}{t}\right), \\ \frac{\partial^2 g}{\partial x^2} &= \left(-\frac{2k}{t^{\frac{3}{2}}} + \frac{4k^2x^2}{t^{\frac{5}{2}}}\right)\exp\left(-\frac{kx^2}{t}\right). \end{aligned}$$

For the equality $\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2}$ to hold, it must be that $k = \frac{1}{4}$, and in this case indeed the equation is satisfied.

Matriculation: ....

4. Compute the integral

$$\iiint\limits_T \frac{z}{1+x^2+y^2} dxdydz,$$

where $T = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq (z-2)^2,\, 0 \leq z \leq 1\}$.
Solution. Going to the cylindrical coordinates, $x = r\cos\theta, y = r\sin\theta, z = z$, the region T corresponds to

$$\widetilde{T} = \{(r, \theta, z) : r^2 \leq (z-2)^2, 0 \leq z \leq 1\} = \{(r, \theta, z) : 0 \leq r \leq 2-z, 0 \leq z \leq 1\}.$$

With the Jacobian $J(r, \theta, z) = r$, we compute the integral:

$$\begin{aligned}
\iiint\limits_T \frac{z}{1+x^2+y^2} dxdydz &= \iiint\limits_{\widetilde{T}} \frac{z}{1+r^2} \cdot r dr d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \int_0^{2-z} \frac{zr}{1+r^2} dr d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \frac{z}{2} \left[\log(1+r^2)\right]_0^{2-z} d\theta dz \\
&= \int_0^1 \int_0^{2\pi} \frac{z}{2} \log(1+(z-2)^2)\, d\theta dz \\
&= \pi \int_0^1 z \log(1+(z-2)^2)\, dz \\
&= \pi \int_0^1 ((z-2)+2) \log(1+(z-2)^2)\, dz
\end{aligned}$$

And the first term can be interated by substitution $u = z-2$ and by noting that $\frac{d}{du}(u^2) = 2u$ and $\int \log(1+t)dt = (1+t)\log(1+t) - t$:

$$\begin{aligned}
\pi \int_0^1 (z-2)\log(1+(z-2)^2)\, dz &= \pi \int_{-2}^{-1} \frac{1}{2} \cdot 2u \log(1+u^2)\, dz \\
&= \frac{\pi}{2} \left[(1+u^2)\log(1+u^2) - u^2\right]_{-2}^{-1} \\
&= \frac{\pi}{2} \left[(2\log 2 - 1) - (5\log 5 - 4)\right]
\end{aligned}$$

As for the second term, by integrating by parts, we have

$$\int_0^1 2\log(1+(z-2)^2)\, dz = 2\left[(z-2)\log(1+(z-2)^2)\right]_0^1 - 2\int_0^1 \frac{2(z-2)^2}{1+(z-2)^2} dz$$

and

$$\begin{aligned}2\int_0^1 \frac{2(z-2)^2}{1+(z-2)^2}dz &= 4\int_0^1 \left(1-\frac{1}{1+(z-2)^2}\right)dz\\ &= 4[z-\arctan(z-2)]_0^1\\ &= 4((1+\tfrac{\pi}{4})-(0-\arctan(-2))\\ &= 4(1+\tfrac{\pi}{4}-\arctan 2)).\end{aligned}$$

Altogether,

$$\begin{aligned}&\iiint\limits_T \frac{z}{1+x^2+y^2}dxdydz\\ &= \pi((\log 2-\tfrac{1}{2})-(\tfrac{5}{2}\log 5-2)+2(-\log 2-(-2)\log 5)-4(1+\tfrac{\pi}{4}-\arctan 2))\\ &= \pi(4\arctan 2-\tfrac{5}{2}-\pi-\log 2+\tfrac{3}{2}\log 5).\end{aligned}$$

($\arctan 2 \approx 1.107..$)

Matriculation: ....

5. Let $\mathbb{F}(x, y, z) = (x(y^2+1), y(z^2+1), z(x^2+1))$ be a vector field on $\mathbb{R}^3$, S be the sphere

$$\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

and let $\mathbb{n}$ be the outgoing normal unit vector on S at each point of S. Compute the surface integral

$$\iint\limits_S \mathbb{F} \cdot \mathbb{n} \, dS.$$

Solution. By Gauss' theorem, the surface integral is equal to

$$\iint\limits_T \operatorname{div} \mathbb{F} \, dxdydz,$$

where $T = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ is the interior of the sphere. We have $\operatorname{div} \mathbb{F}(x, y, z) = (x(y^2+1), y(z^2+1), z(x^2+1)) = (y^2+1) + (z^2+1) + (x^2+1)$.

Going to the spherical coordinates, T corresponds to $\widetilde{T} = \{(r, \theta, \varphi) : r^2 \leq 1\}$ and we can compute the integral:

$$\begin{aligned}
\iint\limits_T \operatorname{div}\mathbb{F} \, dxdydz &= \iint\limits_{\widetilde{T}} \operatorname{div} \mathbb{F} \, r^2 \sin\theta \, drd\theta d\varphi \\
&= \int_0^\pi \int_0^{2\pi} \int_0^1 (r^2+3) r^2 \sin\varphi \, drd\theta d\varphi \\
&= \int_0^\pi \int_0^{2\pi} \int_0^1 \left[\frac{r^5}{5} + r^3\right]_0^1 \sin\theta \, drd\theta d\varphi \\
&= 4\pi \cdot \frac{6}{5} \\
&= \frac{24\pi}{5}.
\end{aligned}$$