Unitary modules and conformal nets associated with the $\mathcal{W}_3$-algebra with $c \geq 2$

Yoh Tanimoto

University of Rome "Tor Vergata"

Kraków-Poznań-Warsaw mathematical physics seminar
19 November 2021

Two-dimensional conformal field theory

- Fields in two-dimensional conformal field theories have often very specific algebraic relations. Heisenberg, Virasoro, Kac-Moody, $\mathcal{W}_N$...
- To have a "unitary" QFT, these field must admit a **unitary** representation.
- To have a conformal Haag-Kastler net, such fields must commute strongly (cf. commutator theorem).

Main results

- The $\mathcal{W}_3$-algebra: an extension of the Virasoro algebra (the stress-energy tensor)
- Unitarity of certain lowest weight representations, including the vacuum representation.
- New technique for strong commutativity for conformal fields, construction of conformal nets for the $\mathcal{W}_3$-algebras.

2d CFT

Two-dimensional conformal symmetry: $\mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R})$. Some important observables (the stress-energy tensor, currents...) decompose into the "chiral components", and they depend only on the lightray variables $x_+, x_- \in \mathbb{R}$.

These observables extend to S^1 by Möbius symmetry, and often considered as fields $\phi(z)$ with variable $z \in S^1 \subset \mathbb{C}$.

A **conformal net** on S^1 is a map $\mathcal{A}$ from the family of intervals in S^1 into the family of von Neumann algebras on $\mathcal{H}$ which satisfies

- Isotony: $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: $I \cap J \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$.
- Conformal covariance: $\exists U$: positive energy projective representations of $\mathrm{Diff}(S^1)$ such that $\mathrm{Ad}\, U(g)\mathcal{A}(I) = \mathcal{A}(gI)$.
- Vacuum: $\exists \Omega$ such that $U(g)\Omega = \Omega$ for $g \in$ Möb and cyclic for $\mathcal{A}(I)$.

The Virasoro algebra

The stress-energy tensor $L(f) = \sum_n \hat{f}_n L_n$ satisfies the commutation relations of the vector fields (plus a central extension)

$$[L(f), L(g)] = L(f'g - fg') + \frac{c}{24} \int (f'''(z) - f'(z))g(z)dz$$

The Virasoro algebra is generated by $\{L_n : n \in \mathbb{Z}, C\}$ with

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}m(m^2 - 1)\delta_{m+n,0},$$

where C is a central element.

cf. $[L_m, L_n] = (m - n)L_{m+n}$ for $m, n = -1, 0, 1$, the Möbius group (translation, dilation and special conformal transformations).

The Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}m(m^2 - 1)\delta_{m+n,0},$$

This is an infinite-dimensional Lie algebra. One can construct **lowest weight representations (modules)** parametrized by $c, h \in \mathbb{R}$, where there is a vector Ω such that $L_n\Omega = 0$ for $n > 0$, $L_0\Omega = h\Omega$, $C\Omega = c\Omega$, and spanned by vectors of the form $L_{-n_1} \cdots L_{-n_k}\Omega$, $n_j > 0$. This is equipped with an invariant sesquilinear form $\langle \cdot, \cdot \rangle$, with respect to which $L_n^* = L_{-n}$.

One considers the field $L(z) = \sum_n L_n z^{-n-2}$ in the "vacuum representation" $h = 0, c \in \mathbb{R}$. Then this alone generates a conformal net (the Virasoro net): $L(f)$ and $L(g)$ commute strongly because we can apply the commutator theorem with L_0.

The Virasoro algebra: question of unitarity

Unitarity: the invariant sesquilinear form is positive semi-definite.

Unitary lowest weight representations are

- discrete series $c = 1 - \frac{6}{m(m+1)}, m = 2, 3, 4, \cdots,$
  $h = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, r = 1, 2, \cdots, m - 1, s = 1, 2, \cdots, r$
- continuous region $c \geq 1, h \geq 0$.

This is proven by

- constructing concrete unitary representations, by embedding the Virasoro algebra into some larger algebra (Goddard-Kent-Olive)
- computing the determinant of the Gram matrix on each subspace spanned by $L_{-n_1} \cdots L_{-n_k} \Omega$ with fixed $n = \sum n_j$, (Kac determinant formula, Feigin-Fuchs)
- for $c \geq 1, h \geq 0$, it is enough that there is one unitary representation
- proving that other values of c, h give non-unitary representations (Friedan-Qiu-Shenker)

The $\mathcal{W}_3$-algebra

A **non-Lie algebraic** extension of the Virasoro algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}m(m^2 - 1)\delta_{m+n,0},$$

$$[L_m, W_n] = (2m - n)W_{m+n},$$

$$[W_m, W_n] = \frac{C}{3 \cdot 5!}(m^2 - 4)(m^2 - 1)m\delta_{m+n,0}$$
$$+ b^2(m - n)\Lambda_{m+n} + \frac{1}{20}(m - n)(2m^2 - mn + 2n^2 - 8)L_{m+n},$$

where $\Lambda_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq-2} L_k L_{n-k} - \frac{3}{10}(n + 2)(n + 3)L_n$ and $b^2 = \frac{16}{22+5C}$.

The lowest weight representations ($L_n\Omega = W_n\Omega = 0$ for $n > 0$, $L_0\Omega = h\Omega$, $W_0\Omega = w\Omega$, $C\Omega = c\Omega$) are parametrized by $(c, h, w) \in \mathbb{R}$. If $h = w = 0$ and the representation is unitary, fields can be constructed from $L(f) = \sum_n \hat{f}_n L_n$ and $W(g) = \sum_n \hat{g}_n W_n$ on $C^\infty(L_0)$.

The $\mathcal{W}_3$-field

As fields, they satisfy

$$[L(z), L(\zeta)] = \delta(z - \zeta)\partial_\zeta L(\zeta) + 2\partial_\zeta\delta(z - \zeta)L(\zeta) + \frac{c}{12}\partial_\zeta^3\delta(z - \zeta),$$

$$[L(z), W(\zeta)] = 3\partial_\zeta\delta(z - \zeta)W(\zeta) + \delta(z - \zeta)\partial_\zeta W(\zeta),$$

$$[W(z), W(\zeta)] = \frac{c}{3 \cdot 5!}\partial_\zeta^5\delta(z - \zeta) + \frac{1}{3}\partial_\zeta^3\delta(z - \zeta)L(\zeta) + \frac{1}{2}\partial_\zeta^2\delta(z - \zeta)\partial L(\zeta)$$

$$+ \partial_\zeta\delta(z - \zeta)\left(\frac{3}{10}\partial_\zeta^2 L(\zeta) + 2b^2\Lambda(\zeta)\right)$$

$$+ \delta(z - \zeta)\left(\frac{1}{15}\partial_\zeta^3 L(\zeta) + b^2\partial_\zeta\Lambda(\zeta)\right)$$

where $b^2 = \frac{16}{22+5c}$ and $\Lambda(z) = \; : L(z)^2 : -\frac{3}{10}\partial_z^2 L(z)$.

The $\mathcal{W}_3$-algebra: unitarity of the vacuum representations

- Let a be the derivative of the massless free field (on one chiral component): $[a(z), a(w)] = \partial_w \delta(z - w)$.

- For $c \geq 2$, the $\mathcal{W}_3$-algebra can be realized in the tensor product of two free fields $a_{[1]}, a_{[2]}$ (Fateev-Zamolodchikov), for $\alpha_0 \in \mathbb{C}$:

$$\tilde{L}(z; \alpha_0) = \frac{1}{2} : a_{[1]}(z)^2 : + \frac{1}{2} : a_{[2]}(z)^2 : + \sqrt{2}\alpha_0 \partial a_{[1]}(z),$$

$$\tilde{W}(z; \alpha_0) = \frac{b}{12i}\big[i2\sqrt{2} : a_{[2]}(z)^3 : -i6\sqrt{2} : a_{[1]}(z)^2 : a_{[2]}(z)$$
$$- i6\alpha_0 \partial a_{[1]}(z)a_{[2]}(z) - i18\alpha_0 a_{[1]}(z)\partial a_{[2]}(z)$$
$$- i6\sqrt{2}\alpha_0^2 \partial^2 a_{[2]}(z)\big],$$

where :: represents the normal product.

(checked by a Mathematica package OPEdefs)

The $\mathcal{W}_3$-algebra: restoring the unitarity

- The Fateev-Zamolodchikov representation does not satisfy unitary for $\alpha_0 \neq 0$:

$$(z^3\tilde{W}(z; \alpha_0))^* = z^3\tilde{W}(z; \alpha_0), \quad (z^2\tilde{L}(z; \alpha_0))^* = z^2\tilde{L}(z; \alpha_0)$$

  with respect to the scalar product coming from the Heisenberg algebra $a_{[1]}(z), a_{[2]}(z)$.

- Consider the automorphism
  $a_{[1]}(z) \mapsto a_{[1]}(z) + i\frac{\alpha_0(z-1)}{\sqrt{2}z(z+1)} + \frac{i\alpha_0}{\sqrt{2}z}, \quad a_{[2]}(z) \mapsto a_{[2]}(z)$
  (cf. Buchholz-Schulz=Mirbach).

- By composition, for $\alpha_0 \in \mathbb{R}$ we restore unitarity except the point $z = -1$.

- On the subspace generated from $\Omega_{[1]} \otimes \Omega_{[2]}$, unitarity holds.

The $\mathcal{W}_3$-algebra: restoring the unitarity

> ## Theorem (Carpi-T.-Weiner arXiv:1910.08334, to appear in Transform. Groups)
>
> *The lowest weight representations of the $\mathcal{W}_3$-algebra associated with the values $c \geq 2, h = w = 0$ are unitary.*

We have two conformal fields $W(z), L(z)$ and a vacuum vector.

Next question: do we have a corresponding conformal net?

Remarks: this is not a unitary subalgebra of $a_{[1]}, a_{[2]}$ on S^1. It is a unitary subalgebra on $\mathbb{R}$, but does not satisfy the Haag duality.

The $\mathcal{W}_3$-conformal net

A conformal net is associated with quantum fields if the fields commute strongly (**strong locality**, Carpi-Kawahigashi-Longo-Weiner '18)

- $\mathcal{A}(I) = \{W(f), L(f) : \operatorname{supp} f \subset I\}''$, the von Neumann algebra generated by the polar decomposition, where $W(f) = \sum_n \bar{f}_n W_n$.
- W-field has conformal dimension 3, and does not satisfy the linear energy bound.
- Do $W(f)$ and $W(g)$ commute strongly if f, g have disjoint supports?

Strong commutativity

Nelson's counterexample

- $L^2(X)$, where X is the Riemann surface obtained by glueing two cutted $\mathbb{R}^2$.
- $\mathscr{D}$: the set of smooth functions whose supports do not contain 0
- A be the derivative in x, B the derivative in y.

A and B commute on $\mathscr{D}$, while e^{itA} and e^{isB} are translations on X which do not commute **globally**.
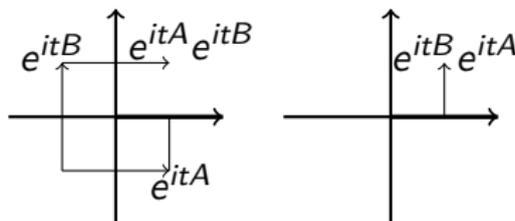
When strong commutativity fails, there is a good reason.

Linear energy bound

- ϕ a Wightman field: for each test function f, $\phi(f)$ is a symmetric operator.
- $[\phi(f), \phi(g)] = 0$ if $\operatorname{supp} f, \operatorname{supp} g$ are spacelike separated (weak locality).
- **Hamiltonian**: $[H, \phi(f)] = i\phi(f')$ (translation covariance).

Linear energy bound

$\|\phi(f)\Psi\| \leq C_f \|(H + r_f \mathbb{1})\Psi\|$ for all f.

In this case, $\|[H, \phi(f)]\Psi\| = \|\phi(f')\Psi\| \leq C_{g'}\|(H + r_{g'}\mathbb{1})\Psi\|$ and one can apply the Driessler-Fröhlich theorem with $T = H$ (Glimm-Jaffe). Many interacting scalar fields (including $\mathscr{P}(\phi)_2$ models) have a corresponding Haag-Kastler net.

Primary fields in 2d CFT

- $W(z) = \sum W_n z^{-n-3}$: primary (diffeomorphism covariant) field on S^1 with conformal dimension 3.
- $L(z) = \sum L_n z^{-n-2}$: Virasoro algebra (Lie algebra of $\mathrm{Diff}(S^1)$).

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0},$$
$$[L_m, W_n] = ((3 - 1)m - n)W_{m+n},$$

(Conformal) Hamiltonian $H = L_0 = L(1)$.

- Bad news: A primary field with dimension $d = 3 > 2$ **never** satisfies linear energy bound.
- Good news: for arbitrariy f, $[W(f^2), L(f)] = 0$.

Can $L(f)$ be used for "local" Hamiltonian?

Local energy bounds

Theorem

*A primary field ϕ with conformal dimension d can satisfy **at best** the following bound:*

$$\|\phi_0 \Psi\| \leq C \|(L_0 + r)^{d-1} \mathbb{1} \Psi\|$$

*If this holds, then it satisfy the following **local energy bound**:*

$$\|\phi(f^{d-1})\Psi\| \leq \tilde{C} \|(L(f) + r\mathbb{1})^{d-1}\Psi\|$$

*for **non-negative** test function f.*

Proof: we have $U(\gamma)\phi(g)U(\gamma)^* = \phi((\gamma' \circ \gamma^{-1})^{d-1}(g \circ \gamma^{-1}))$ for test function f and $\gamma \in \mathrm{Diff}(S^1)$ and $U(\gamma)L_0 U(\gamma)^* = L(\gamma' \circ \gamma^{-1}) + r_\gamma$.
$\gamma' \circ \gamma^{-1} = g$ must satisfy $\int \frac{1}{g} = 2\pi$.
To extend this to general nonnegative f, we need the optimal estimate.

A sufficient condition for strong commutativity

Theorem (Driessler-Fröhlich)

Let T be a positive self-adjoint operator, A, B symmetric operators on $\mathrm{Dom}(T)$ such that for $\Psi, \Phi \in \mathrm{Dom}(T)$

- $\|A\Psi\| \leq C\|T\Psi\|, \|B\Psi\| \leq C\|T\Psi\|$ *for* $\Psi \in \mathrm{Dom}(T)$.
- $|\langle A\Psi, T\Phi \rangle - \langle T\Psi, A\Phi \rangle| \leq C\|T\Psi\|\|\Phi\|$,
  $|\langle B\Psi, T\Phi \rangle - \langle T\Psi, B\Phi \rangle| \leq C\|T\Psi\|\|\Phi\|$.
- $|\langle A\Psi, T\Phi \rangle - \langle T\Psi, A\Phi \rangle| \leq C\|T^{\frac{1}{2}}\Psi\|\|T^{\frac{1}{2}}\Phi\|$,
  $|\langle B\Psi, T\Phi \rangle - \langle T\Psi, B\Phi \rangle| \leq C\|T^{\frac{1}{2}}\Psi\|\|T^{\frac{1}{2}}\Phi\|$.
- $\langle A\Psi, B\Phi \rangle = \langle B\Psi, A\Phi \rangle$

Then A and B strongly commute.

The difficult part is estimating $[H, A], [H, B]$ by T.

The $\mathcal{W}_3$-net

- unitary vacuum representations of $W(z), L(z)$ parametrized by $c \geq 2$.
- W-field has conformal dimension 3.
- W **satisfies the optimal bound** $\|W(f^2)\Psi\| \leq C\|(L(f) + r_f \mathbb{1})^2\Psi\|$.
- $[W(f^2), L(f)] = 0. \implies$ Driessler-Fröhlich theorem with $T = (L(f) + L(g) + r_{f,g})^2$ for nonnegative f, g, to prove that $W(f^2), W(g^2)$ commute strongly.
- The fields of the form $W(f^2)$ are invariant under diffeomorphisms.
- A posteriori we can also prove that $W(f)$ and $W(g), L(g)$ commute strongly.

Theorem

The $\mathcal{W}_3$-algebra for $c \geq 2$ has an associated conformal Haag-Kastler net.

Summary and outlook

- Conformal net $\Rightarrow$ conformal field (Fredenhagen-Jörss, Jörss, Henriques-Tener...)
- Conformal field $\overset{?}{\Rightarrow}$ conformal net
- We need optimal energy bound or local energy bound.

- CFT have charged primary fields and extensions. Their locality could be proven using local energy bounds.
- Perturbation of CFT by primary fields? (joint with C. Jäkel)
- Massive integrable models may have wedge-local fields with different domains of self-adjointness (joint with H. Bostelmann and D. Cadamuro).