Strong locality beyond linear energy bounds

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From a Haag-Kastler net, various subfactors arise. If there is a representation of a conformal net, one can construct the Jones-Wassermann subfactor. Alternatively, if there is an extension or a subtheory, each local algebra gives directly a subfactor. Therefore, while constructing new examples of Haag-Kastler net is an important problem by itself, it is also interesting from the point of view of subfactor.

Most examples are constructed from quantum fields, namely operator-valued distributions. A quantum field $\phi(x)$ (a Wightman field, or a field in a vertex operator algebra satisfying a polynomial energy bound) should satisfy locality: if f, g are test functions with spacelike separated supports, then $\phi(f), \phi(g)$ should commute (on a suitable domain). In order to construct a Haag-Kastler net, we need **strong locality**: $\phi(f), \phi(g)$ should strongly commute (their spectral projections commute). In various examples, strong locality is the last technical barrier to obtain a Haag-Kastler net.

The most commonly used tool to show strong locality is Driessler-Fröhlich theorem: if A and B commute on the domain of a H and satisfies $\|A\Psi\| \leq C\|H\Psi\|$ with $C > 0$ and $[H, A]$ can be estimated as a sesquilinear form by H, and similar estimates hold for B, then A and B strongly commute. When H is the "Hamiltonian" satisfying $[H, \phi(f)] = i\phi(f')$, the bound $\|\phi(f)\Psi\| \leq C\|H\Psi\|$ for all Ψ suffices. This is called a linear energy bound.

Linear energy bound does not apply to some interesting models, yet we can prove strong commutativity by a trick using the Driessler-Fröhlich theorem. Indeed, in this way we construct two new families of Haag-Kastler net: one is a chiral conformal net (the **$\mathcal{W}_3$-algebra**) and the other is a two-dimensional integrable QFT (the **Bullough-Dodd model**).

- The $\mathcal{W}_3$-algebra [3]. This is an extension of the Virasoro algebra as a vertex operator algebra. It is generated by the Virasoro field $L(z)$ and an additional field $W(z)$, satisfying the commutation relations

  $$\begin{aligned}
  [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
  [L_m, W_n] &= (2m-n)W_{m+n}, \\
  [W_m, W_n] &= \frac{c}{3\cdot 5!}(m^2-4)(m^2-1)m\delta_{m+n,0} \\
  &\qquad + b^2(m-n)\Lambda_{m+n} + \left[\frac{1}{20}(m-n)(2m^2-mn+2n^2-8))\right]L_{m+n},
  \end{aligned}$$

  where $\Lambda_n = \sum_{k>-2} L_{n-k}L_k + \sum_{k\leq -2} L_k L_{n-k} - \frac{3}{10}(n+2)(n+3)L_n$ and $c \in \mathbb{C}, c \neq -\frac{22}{5}, b^2 = \frac{16}{22+5c}$. The $W(z)$ field has conformal dimension 3, and from this it follows that it cannot satisfy the linear energy bound, where $H = L_0$ is the conformal Hamiltonian. When the central charge c is larger than 2, then no coset realization is known.

  We proved in [2] that for $c \geq 2$ the vacuum representation is unitary, hence the fields $L(z), W(z)$ are good candidates for quantum fields. The commutator $[W(z), W(w)]$ contains a non-linear expression in $L(z)$, and from this we can have a local energy bound: for $f \geq 0$, $\|W(f^{d-1})\Psi\| \leq C\|(L(f) + c_f \mathbb{1})^{d-1}\|$. Note that, from the commutation relations, we have $[W(f^{d-1}), L(f)] = 0$. Then we can apply the Driessler-Fröhlich theorem with $H = (L(f) + L(g) + c\mathbb{1})^{d-1}$ for non-negative f, g. and obtain strong locality.

- The Bullough-Dodd model. This is a variation of the models of [4], where the S-matrix S contains poles at $\zeta = \frac{\pi i}{3}, \frac{2\pi i}{3}$:

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)} \cdot \frac{\tanh \frac{1}{2}\left(\theta - \left(\frac{\pi}{3} - \epsilon\right) i\right)}{\tanh \frac{1}{2}\left(\theta + \left(\frac{\pi}{3} + \epsilon\right) i\right)} \frac{\tanh \frac{1}{2}\left(\theta - \left(\frac{\pi}{3} + \epsilon\right) i\right)}{\tanh \frac{1}{2}\left(\theta + \left(\frac{\pi}{3} - \epsilon\right) i\right)}$$

  where $0 < \epsilon < \frac{\pi}{6}$. The S-symmetric Fock space is the Fock space based on $L^2(\mathbb{R})$ with the symmetry $\Psi(\theta_1, \theta_2) = S(\theta_2 - \theta_1)\Psi(\theta_2, \theta_1)$. With the (left-)creation and annihilation operator $z^\dagger, z$, one may form an operator $\phi(f) = z^\dagger(f) + z(f)$. This operator is localized in the left wedge if S is analytic [4], but for the S above we need to correct: $\tilde{\phi}(f) = \phi(f) + \chi(f)$, where $\chi(f) = \sum n P_n(\chi_1(f) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1})P_n$, P_n is the projection onto the n-particle space and with $R = \operatorname*{Res}_{\zeta = \frac{2\pi i}{3}} S(\zeta)$

$$(\chi_1(f))\Psi(\theta) = 2\pi\sqrt{|R|} f^+(\theta + \tfrac{\pi i}{3})\Psi(\theta - \tfrac{\pi i}{3})$$

  and f^+ is the one-particle function corresponding to a test function f.

  We find a nice self-adjoint extension of $\chi(f)$. Similarly, $\tilde{\phi}'(g), \chi'(g)$ can be constructed for reflected wedges, Strong commutativity between $\tilde{\phi}(f), \tilde{\phi}'(g)$ can be shown by applying Driessler-Fröhlich theorem to $H = \tilde{\phi}(f) + \tilde{\phi}(g) + C(f, g)N$, where $C(f, g) > 0$ and N is the number operator. For the self-adjointness of H which follows from that of $\chi(f)$, we use a variation of the Kato-Rellich perturbation.

  We also prove the existence of operators in (large enough) double cones, which completes the construction of Haag-Kastler net.

References

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