Towards construction of integrable QFT with bound states

Yoh Tanimoto

Graduate School of Mathematical Sciences, the University of Tokyo,
3-8-1 Komaba, Tokyo, 153-8914, Japan
**E-mail: hoyt@ms.u-tokyo.ac.jp*
www.ms.u-tokyo.ac.jp/˜hoyt/

We review recent operator-algebraic constructions of quantum field theories, especially of two-dimensional integrable models. In the operator-algebraic approach, a model of quantum field theory is realized as a net of von Neumann algebras associated to spacetime regions. A key idea in the recent developments is to construct first the observables localized in wedge-shaped regions, then to define the algebras for double cones by intersection.

Up to now, these constructions are limited to the class of S-matrices whose components are analytic in rapidity in the physical strip. We present candidates for observables in wedge regions for scalar factorizing S-matrices with poles in the physical strip. We discuss the self-adjointness of these candidate operators and strong commutativity between them.

Keywords: Quantum field theory; operator algebras; integrable models; scattering theory; Rindler wedge.

1. Operator-algebraic approach to QFT

A mathematically consistent construction of Quantum Field Theories (QFT) is a hard task: On the one hand, the conventional perturbation theory has been phenomenologically very successfully and is well-understood as formal power series.[3] On the other hand, it has been argued that Quantum Electrodynamics, a representative QFT, has divergent perturbative series in coupling constant.[7]

A program called Constructive QFT[8] has been quite successful in constructing models of QFT in two- and three-dimensional spacetimes. These models fit into mathematically sound frameworks (Wightman, Osterwalder-Schrader and Haag-Kastler axioms) and are shown to be different from the physically trivial free theories. However, no nontrivial model has been constructed in the physical four dimensions.

Several progresses have been made in the last decade in the operator-algebraic framework (**Haag-Kastler nets**). A central idea is that, while an interacting local quantum fields can be very complicated objects, an infinitely-extended spacetime region can accommodate observables which have moderate behaviors in the momentum space. Especially, Schroer proposed the use of so-called wedge-local fields for a class of integrable QFT in two dimensions.[14] Lechner took these wedge-local fields as a starting point and constructed Haag-Kastler nets for a family of integrable QFT.[11] See Ref. 12 a more detailed review.

This program has been extended to several directions. Models with multiple particle spectrum have been also studied,[13] but the existence proof of local observables is at the moment still missing.[1] Given a model with global gauge symmetry, a purely operator-algebraic method to obtain new models has been developed.[16] Relations between massless models and conformal field theories have been also investigated.[2,15] The novel result in this review is some progress in models with bound states, or with S-matrices which have poles in the physical strip.[5]

2. Haag-Kastler nets

In the traditional axiomatic QFT, one considers a Wightman field ϕ, which is an operator-valued distribution ϕ on the spacetime $\mathbb{R}^d$. It represents pointwise localized observables. For a spacetime region O, one takes the algebra of operators $\mathcal{A}(O) := \{e^{i\phi(f)} : \mathrm{supp} f \subset O\}''$, where $\mathcal{M}'$ means the set of bounded operators commuting with any element of $\mathcal{M}$. The double commutant $\mathcal{M}''$ is the smallest von Neumann algebra which includes $\mathcal{M}$. They are the algebra of observables which can be measured in the region O.

More in general, a **Haag-Kastler net**, or a **Poincaré covariant net (of observables)** is an assignment of a von Neumann algebra $\mathcal{A}(O)$ to each open region O. It should be covariant with respect to a continuous unitary representation U of the Poincaré group on $\mathcal{H}$ and there should be an invariant ground state, the vacuum Ω. A set of these objects encodes the full information of the given model and one can, for example, define canonically the scattering operator under some assumptions on U.[9]

A fundamental open problem can now be stated as follows: to construct examples of interacting nets in physical four spacetime dimensions. For two and three dimensions, several families of interacting nets have been constructed, including those with polynomial interaction and especially the ϕ^4_3 model in three dimensions.

3. Wedge-local net of observables

Recently, we have seen several constructions of Haag-Kastler nets in two dimensions which are not (directly) generated by Wightman fields. In general, interacting Wightman fields are not easy objects to construct. Instead, if one considers observables localized in infinitely extended regions, there is a better chance to find more tractable observables. Thereafter, the algebras for compact regions can be defined as the intersection of the algebras of such extended regions. This idea has been proposed by Schroer for so-called integrable QFT[14] and implemented by Lechner.[11]

More precisely, one considers the (left) Rindler wedge: $W_{\mathrm{L}} = \{(a_0, a_1) : -a_1 > |a_0|\}$. In two dimensions, any double cone is the intersection of two translated wedges: $D_{a,b} = (W_{\mathrm{L}} + a) \cap (W_{\mathrm{R}} + b)$. So the strategy is first to find appropriate observables $\{\phi\}$ which should be localized in W_{L} then to take the von Neumann algebra $\mathcal{M}$ generated by them. One has also to have the spacetime symmetry U. Finally, the net is defined first for double cones by $\mathcal{A}(D_{a,b}) := U(a)\mathcal{M}U(a)^* \cap U(b)\mathcal{M}'U(b)^*$, and for an arbitrary region O by $\mathcal{A}(O) := \bigcup_{D_{a,b} \subset O} A(D_{a,b})$.

Two-dimensional spacetime is special because a double cone is obtained as the intersection of two wedges. Thanks to this property, one can prove that the abstractly defined algebra $\mathcal{A}(D_{a,b})$ is nontrivial for some integrable models by examining a property of Lorentz boosts called **modular nuclearity**.[4]

4. Integrable models

Let us consider a particular class of models which contains only one species of particle, in which the number of particles is preserved during the scattering process. By the conservation of energy and momentum, the two-particle scattering process is given by a phase $S(\theta)$, where θ is the difference of rapidity of two particles. Furthermore, one may assume that the higher-particle scattering operator factorizes[6].

First the Hilbert space is constructed as follows: the one-particle space is $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$ and n-particle space $\mathcal{H}_n$ consists of the S-symmetric functions:

$$\Psi_n(\theta_1, \cdots, \theta_n) = S(\theta_{k+1} - \theta_k)\Psi_n(\theta_1, \cdots, \theta_{k+1}, \theta_k, \cdots, \theta_n).$$

On the S-symmetric Fock space $\mathcal{H} = \bigoplus_n \mathcal{H}_n$, one can define the creation and annihilation operators $z^\dagger, z$ as in the usual Fock space. Lechner proved that, if S has no pole in the physical strip $\mathbb{R} + i(0, \pi)$, then for real test functions f supported in W_{L},

$$\phi(f) := z^\dagger(f^+) + z(f^+), \quad f^+(\theta) := \int da e^{ip(a)\cdot a} f(a), \quad p(\theta) = (m\cosh\theta, m\sinh\theta),$$

where $m > 0$ is the mass of the particle, generate the algebra $\mathcal{A}(W_{\mathrm{L}})$ of the wedge.[10] Furthermore, if S satisfies a certain regularity condition, the local algebra $\mathcal{A}(O)$ is nontrivial and the two-particle scattering function turns out to be S.[11]

This construction of observables in wedges can be generalized to models with more complicated particle spectrum.[13] In this general case, the existence proof of local observables works partially,[1] but the fact that S satisfies nontrivial Yang-Baxter equation makes some estimates of the boost operator more complicated.

5. Models with bound states

The cases where S has poles turned out to be much more difficult than expected. The poles in S should correspond to bound states of elementary particles, therefore, the model should represent more complicated scattering processes.

Let us consider the simplest possible case, where there is still only one species of particle. Then the poles of S must appear only at $\frac{\pi i}{3}, \frac{2\pi i}{3}$. In 5, we introduced an operator $\chi(f) = \bigoplus_n \chi_n(f)$ by

$$(\chi_1(f)\Psi_1) = \sqrt{2\pi|R|} f^+\left(\theta + \frac{\pi}{3}\right)\Psi_1\left(\theta - \frac{\pi}{3}\right), \quad \chi_n(f) = P_n(\chi_1(f)\otimes I \otimes\cdots\otimes I)P_n,$$

where P_n is the projection from $\mathcal{H}_1^{\otimes n}$ onto the space $\mathcal{H}_n$ of S-symmetric functions and $R = \mathrm{Res}_{\zeta=\frac{2\pi i}{3}} S(\zeta)$. If we set $\widetilde{\phi}(f) = \phi(f) + \chi(f)$ and similarly $\widetilde{\phi}'(g)$ for the right wedge W_{R}, we can show that they commute weakly on a dense domain.

In contrast to $\phi(f)$, $\chi(f)$ has a subtle domain property and it is at the moment unclear whether $\widetilde{\phi}(f)$ and $\widetilde{\phi}'(g)$ strongly commute. Yet, once this is established, we are expecting that also the modular nuclearity can be proved.

Acknowledgments

This work was supported by Grant-in-Aid for JSPS fellows 25-205.

References

1. Sabina Alazzawi. Deformations of quantum field theories and the construction of interacting models. 2014. Ph.D. thesis, Universität Wien. Available at http://arxiv.org/abs/1503.00897.
2. Marcel Bischoff and Yoh Tanimoto. Integrable QFT and Longo-Witten endomorphisms. *Ann. Henri Poincaré*, 16(2):569–608, 2015.
3. R. Brunetti, M. Dütsch, and K. Fredenhagen. Perturbative algebraic quantum field theory and the renormalization groups. *Adv. Theor. Math. Phys.*, 13(5):1541–1599, 2009.
4. Detlev Buchholz and Gandalf Lechner. Modular nuclearity and localization. *Ann. Henri Poincaré*, 5(6):1065–1080, 2004.
5. Daniela Cadamuro and Yoh Tanimoto. Wedge-local fields in integrable models with bound states. arXiv:1502:01313, to appear in *Comm. Math. Phys.*
6. Patrick Dorey. Exact S-matrices. In *Conformal field theories and integrable models (Budapest, 1996)*, pages 85–125. Springer, Berlin, 1997.
7. F. J. Dyson. Divergence of perturbation theory in quantum electrodynamics. *Physical Rev. (2)*, 85:631–632, 1952.
8. James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987.
9. Rudolf Haag. *Local quantum physics*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1996.
10. Gandalf Lechner. Polarization-free quantum fields and interaction. *Lett. Math. Phys.*, 64(2):137–154, 2003.
11. Gandalf Lechner. Construction of quantum field theories with factorizing S-matrices. *Comm. Math. Phys.*, 277(3):821–860, 2008.
12. Gandalf Lechner. Algebraic constructive quantum field theory: Integrable models and deformation techniques. arXiv:1503.03822
13. Gandalf Lechner and Christian Schützenhofer. Towards an operator-algebraic construction of integrable global gauge theories. *Ann. Henri Poincaré*, 15(4):645–678, 2014.
14. Bert Schroer. Modular localization and the bootstrap-formfactor program. *Nuclear Phys. B*, 499(3):547–568, 1997.
15. Yoh Tanimoto. Construction of Wedge-Local Nets of Observables Through Longo-Witten Endomorphisms. *Comm. Math. Phys.*, 314(2):443–469, 2012.
16. Yoh Tanimoto. Construction of two-dimensional quantum field models through Longo-Witten endomorphisms. *Forum of Mathematics, Sigma*, 2:e7, 2014.